CASE STUDY 2c Hosts, Parasites, and Time-Travel

In this part of the case study you will formulate a mathematical model for the antagonistic interactions between *Daphnia* and its parasite using differential equations. Let's suppose that there are two possible host genotypes (A and a) and two possible parasite genotypes (B and b). Parasites of type B can infect only hosts of type A, while parasites of type b can infect only hosts of type a (see Table 1). We will derive a set of two coupled differential equations that model the dynamics of the frequency of A in the host population and B in the parasite population.

Table 1
The outcome of challenges between different host and parasite genotypes.

	Host A	Host a
Parasite B	Infection occurs	Infection does not occur
Parasite b	Infection does not occur	Infection occurs

A common differential equation used in biology to model the frequency dynamics of a particular genotype is

BB **(1)**
$$\frac{df}{dt} = f(1 - f)(r_1 - r_2)$$

where f is the frequency of type 1, and r_1 and r_2 are the per capita reproduction rates of the two types. For example, see Exercise 7.2.16. We will use an equation of this form for both the host and the parasite populations.

Suppose the per capita reproduction rate of uninfected hosts is r_q and that for infected hosts is $r_q - s_q$. The constant s_q is assumed to satisfy the inequality $0 < s_q < r_q$ and represents the reduction in reproductive output of a host due to infection. Similarly, the per capita reproduction rate of a parasite that is able to infect a host is r_p and that for one unable to infect a host is $r_p - s_p$ (the parasite can reproduce in the absence of the host, but it does so less well). The constant s_p is assumed to satisfy the inequality $0 < s_p < r_p$ and represents the reduction in reproductive output of a parasite if it is unable to infect a host.

Let's use q to denote the frequency of type A individuals in the host population and p to denote the frequency of type B individuals in the parasite population. Suppose that host–parasite encounters occur at random with respect to genotype.

1. With random encounters, the average per capita reproduction rate for hosts of a given type is $r_B p + r_b(1 - p)$, where r_B and r_b are the reproduction rates of the host when encountering a type B or type b parasite, respectively. Show that the average per capita reproduction rates of hosts of type A and a are therefore

   type A: $\quad r_q - ps_q$

   type a: $\quad r_q - (1 - p)s_q$

2. With random encounters, the average per capita reproduction rate for parasites of a given type is $r_A q + r_a(1 - q)$, where r_A and r_a are the reproduction rates of the parasite when encountering a type A or type a host, respectively. Show that the average per capita reproduction rates of parasites of type B and b are therefore

   type B: $\quad r_p - (1 - q)s_p$

   type b: $\quad r_p - qs_p$

3. Suppose both q and p satisfy differential equations of the form given in Equation 1. Show that q and p therefore satisfy

$$\frac{dq}{dt} = s_q q(1 - q)(1 - 2p)$$

$$\frac{dp}{dt} = s_p p(1 - p)(2q - 1)$$

4. Construct the phase plane including all nullclines, equilibria, and arrows indication the direction of movement in the plane.

5. Explain, qualitatively, how the frequencies of the two parasite genotypes are predicted to change over time. Similarly, explain how the frequencies of the two host genotypes are predicted to change over time.