4.5 THE SUBSTITUTION RULE

EXAMPLE A Find $\int \sqrt{1 + x^2}\, x^5\, dx$.

SOLUTION An appropriate substitution becomes more obvious if we factor x^5 as $x^4 \cdot x$. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so $x\,dx = du/2$. Also $x^2 = u - 1$, so $x^4 = (u - 1)^2$:

$$\begin{aligned}\int \sqrt{1 + x^2}\, x^5\, dx &= \int \sqrt{1 + x^2}\, x^4 \cdot x\, dx \\ &= \int \sqrt{u}\,(u - 1)^2 \frac{du}{2} = \tfrac{1}{2}\int \sqrt{u}\,(u^2 - 2u + 1)\, du \\ &= \tfrac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du \\ &= \tfrac{1}{2}\left(\tfrac{2}{7}u^{7/2} - 2 \cdot \tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right) + C \\ &= \tfrac{1}{7}(1 + x^2)^{7/2} - \tfrac{2}{5}(1 + x^2)^{5/2} + \tfrac{1}{3}(1 + x^2)^{3/2} + C\end{aligned}$$

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EXAMPLE B Evaluate $\int_0^4 \sqrt{2x + 1}\, dx$ using (6).

SOLUTION Using the substitution from Solution 1 of Example 2, we have $u = 2x + 1$ and $dx = du/2$. To find the new limits of integration we note that

when $x = 0$, $u = 2(0) + 1 = 1$ and when $x = 4$, $u = 2(4) + 1 = 9$

Therefore

$$\begin{aligned}\int_0^4 \sqrt{2x + 1}\, dx &= \int_1^9 \tfrac{1}{2}\sqrt{u}\, du \\ &= \tfrac{1}{2} \cdot \tfrac{2}{3}u^{3/2}\Big]_1^9 \\ &= \tfrac{1}{3}(9^{3/2} - 1^{3/2}) = \tfrac{26}{3}\end{aligned}$$

Observe that when using (6) we do not return to the variable x after integrating. We simply evaluate the expression in u between the appropriate values of u. ■

▪ The geometric interpretation of Example B is shown in Figure 1. The substitution $u = 2x + 1$ stretches the interval $[0, 4]$ by a factor of 2 and translates it to the right by 1 unit. The Substitution Rule shows that the two areas are equal.

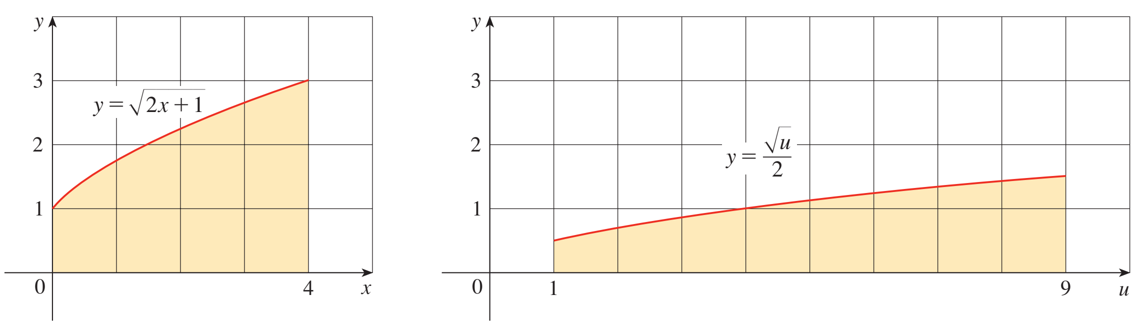

FIGURE 1