6.6 IMPROPER INTEGRALS

EXAMPLE A Evaluate $\int_0^1 \ln x\,dx$.

SOLUTION We know that the function $f(x) = \ln x$ has a vertical asymptote at 0 since $\lim_{x\to 0^+} \ln x = -\infty$. Thus, the given integral is improper and we have

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} \int_t^1 \ln x\,dx$$

Now we integrate by parts with $u = \ln x$, $dv = dx$, $du = dx/x$, and $v = x$:

$$\int_t^1 \ln x\,dx = x\ln x\Big]_t^1 - \int_t^1 dx$$

$$= 1\ln 1 - t\ln t - (1 - t)$$

$$= -t\ln t - 1 + t$$

To find the limit of the first term we use l'Hospital's Rule:

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} = \lim_{t\to 0^+} \frac{1/t}{-1/t^2}$$

$$= \lim_{t\to 0^+} (-t) = 0$$

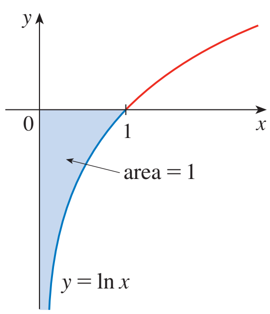

FIGURE 1

Therefore

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+} (-t\ln t - 1 + t)$$

$$= -0 - 1 + 0 = -1$$

Figure 1 shows the geometric interpretation of this result. The area of the shaded region above $y = \ln x$ and below the x-axis is 1. ■