7.6 DIFFERENTIAL EQUATIONS

EXAMPLE A Solve the initial-value problem $xy' = -y$, $x > 0$, $y(4) = 2$.

SOLUTION We write the differential equation as

$$x\frac{dy}{dx} = -y \qquad \text{or} \qquad \frac{dy}{y} = -\frac{dx}{x}$$

Therefore

$$\int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\ln|y| = -\ln|x| + C$$

$$|y| = \frac{1}{|x|}e^C$$

$$y = \frac{K}{x}$$

where $K = \pm e^C$ is a constant. To determine K we put $x = 4$ and $y = 2$ in this equation:

$$2 = \frac{K}{4} \qquad k = 8$$

The solution of the initial-value problem is

$$y = \frac{8}{x} \qquad x > 0$$

Figure 1 shows the family of solutions $xy = K$ for several values of K (equilateral hyperbolas) and, in particular, the solution that satisfies $y(4) = 2$ [the hyperbola that passes through the point (4, 2)]. ■

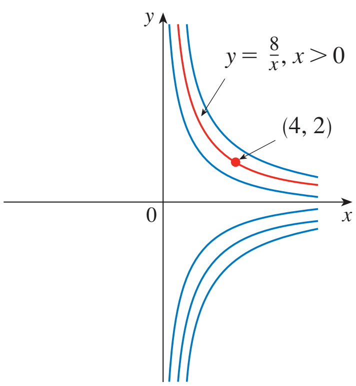

FIGURE 1

EXAMPLE B Solve $y' = 1 + y^2 - 2x - 2xy^2$, $y(0) = 0$, and graph the solution.

SOLUTION At first glance this does not look like a separable equation, but notice that it is possible to factor the right side as the product of a function of x and a function of y as follows:

$$\frac{dy}{dx} = 1 + y^2 - 2x(1 + y^2) = (1 - 2x)(1 + y^2)$$

$$\int \frac{dy}{1 + y^2} = \int (1 - 2x)\,dx$$

$$\tan^{-1}y = x - x^2 + C$$

Putting $x = 0$ and $y = 0$, we get $C = \tan^{-1}0 = 0$, so

$$\tan^{-1}y = x - x^2$$

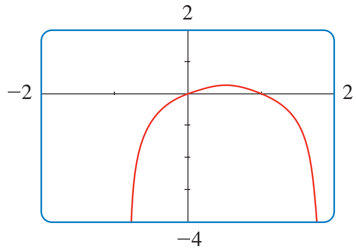

FIGURE 2

To graph this equation we notice that it is equivalent to

$$y = \tan(x - x^2)$$

provided that $-\pi/2 < x - x^2 < \pi/2$. Solving these inequalities using the quadratic formula, we find that

$$\tfrac{1}{2}\left(1 - \sqrt{1 + 2\pi}\right) < x < \tfrac{1}{2}\left(1 + \sqrt{1 + 2\pi}\right)$$

This enables us to graph the solution in Figure 2. ■