1.2 A CATALOG OF ESSENTIAL FUNCTIONS

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V EXAMPLE A Table 1 lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2002. Use the data in Table 1 to find a model for the carbon dioxide level.

SOLUTION We use the data in Table 1 to make the scatter plot in Figure 1, where t represents time (in years) and C represents the CO_2 level (in parts per million, ppm).

TABLE 1

Year	CO_2 level (in ppm)	Year	CO_2 level (in ppm)
1980	338.7	1992	356.4
1982	341.1	1994	358.9
1984	344.4	1996	362.6
1986	347.2	1998	366.6
1988	351.5	2000	369.4
1990	354.2	2002	372.9

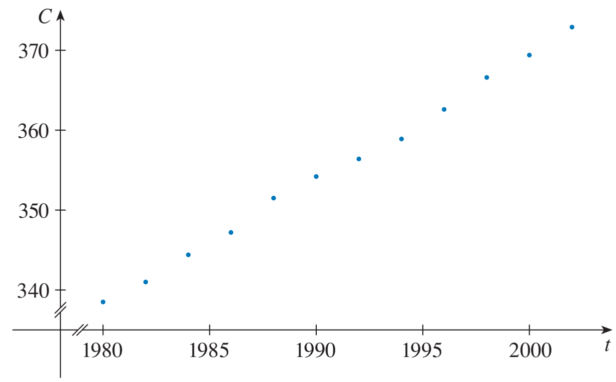

FIGURE 1 Scatter plot for the average CO_2 level

Notice that the data points appear to lie close to a straight line, so it's natural to choose a linear model in this case. But there are many possible lines that approximate these data points, so which one should we use? From the graph, it appears that one possibility is the line that passes through the first and last data points. The slope of this line is

$$\frac{372.9 - 338.7}{2002 - 1980} = \frac{34.2}{22} \approx 1.5545$$

and its equation is

$$C - 338.7 = 1.5545(t - 1980)$$

or

$$C = 1.5545t - 2739.21 \tag{1}$$

Equation 1 gives one possible linear model for the carbon dioxide level; it is graphed in Figure 2.

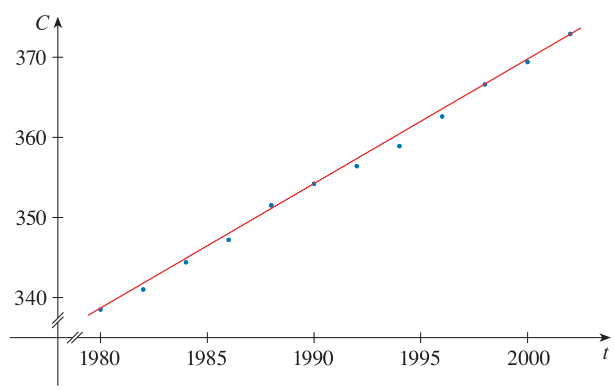

FIGURE 2 Linear model through first and last data points

▪ A computer or graphing calculator finds the regression line by the method of **least squares**, which is to minimize the sum of the squares of the vertical distances between the data points and the line.

Although our model fits the data reasonably well, it gives values higher than most of the actual CO_2 levels. A better linear model is obtained by a procedure from statistics called *linear regression.* If we use a graphing calculator, we enter the data from Table 1 into the data editor and choose the linear regression command. (With Maple we use the fit[leastsquare] command in the stats package; with Mathematica we use the Fit command.) The machine gives the slope and y-intercept of the regression line as

$$m = 1.55192 \qquad b = -2734.55$$

So our least squares model for the CO_2 level is

2 $$C = 1.55192t - 2734.55$$

In Figure 3 we graph the regression line as well as the data points. Comparing with Figure 2, we see that it gives a better fit than our previous linear model.

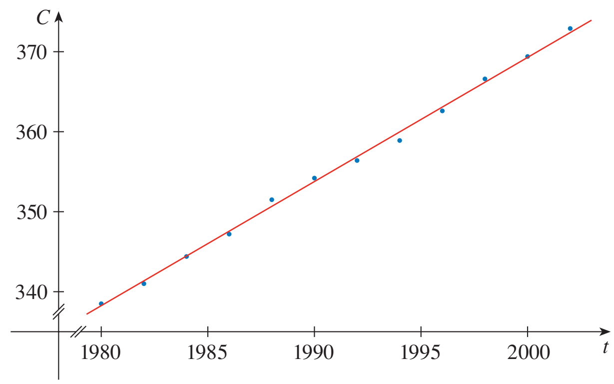

FIGURE 3
The regression line

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EXAMPLE B Use the linear model given by Equation 2 to estimate the average CO_2 level for 1987 and to predict the level for the year 2010. According to this model, when will the CO_2 level exceed 400 parts per million?

SOLUTION Using Equation 2 with $t = 1987$, we estimate that the average CO_2 level in 1987 was

$$C(1987) = (1.55192)(1987) - 2734.55 \approx 349.12$$

This is an example of *interpolation* because we have estimated a value *between* observed values. (In fact, the Mauna Loa Observatory reported that the average CO_2 level in 1987 was 348.93 ppm, so our estimate is quite accurate.)

With $t = 2010$, we get

$$C(2010) = (1.55192)(2010) - 2734.55 \approx 384.81$$

So we predict that the average CO_2 level in the year 2010 will be 384.8 ppm. This is an example of *extrapolation* because we have predicted a value *outside* the region of observations. Consequently, we are far less certain about the accuracy of our prediction.

Using Equation 2, we see that the CO_2 level exceeds 400 ppm when

$$1.55192t - 2734.55 > 400$$

Solving this inequality, we get

$$t > \frac{3134.55}{1.55192} \approx 2019.79$$

We therefore predict that the CO_2 level will exceed 400 ppm by the year 2019. This prediction is somewhat risky because it involves a time quite remote from our observations. ■

TABLE 2

Time (seconds)	Height (meters)
0	450
1	445
2	431
3	408
4	375
5	332
6	279
7	216
8	143
9	61

EXAMPLE C A ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground, and its height h above the ground is recorded at 1-second intervals in Table 2. Find a model to fit the data and use the model to predict the time at which the ball hits the ground.

SOLUTION We draw a scatter plot of the data in Figure 4 and observe that a linear model is inappropriate. But it looks as if the data points might lie on a parabola, so we try a quadratic model instead. Using a graphing calculator or computer algebra system (which uses the least squares method), we obtain the following quadratic model:

$$h = 449.36 + 0.96t - 4.90t^2 \tag{3}$$

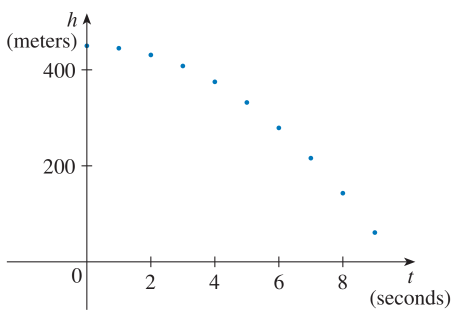


FIGURE 4
Scatter plot for a falling ball

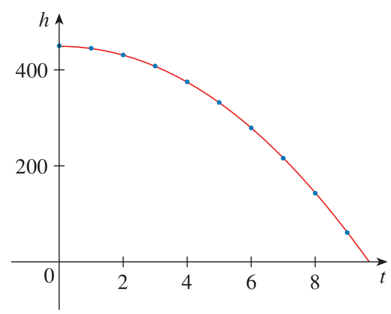


FIGURE 5
Quadratic model for a falling ball

In Figure 5 we plot the graph of Equation 3 together with the data points and see that the quadratic model gives a very good fit.

The ball hits the ground when $h = 0$, so we solve the quadratic equation

$$-4.90t^2 + 0.96t + 449.36 = 0$$

The quadratic formula gives

$$t = \frac{-0.96 \pm \sqrt{(0.96)^2 - 4(-4.90)(449.36)}}{2(-4.90)}$$

The positive root is $t \approx 9.67$, so we predict that the ball will hit the ground after about 9.7 seconds. ■

EXAMPLE D Sketch the graph of the function $f(x) = x^2 + 6x + 10$.

SOLUTION Completing the square, we write the equation of the graph as

$$y = x^2 + 6x + 10 = (x + 3)^2 + 1$$

This means we obtain the desired graph by starting with the parabola $y = x^2$ and shifting 3 units to the left and then 1 unit upward (see Figure 6).

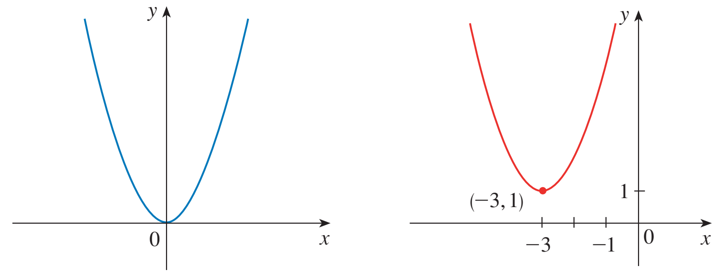


FIGURE 6 (a) $y = x^2$ (b) $y = (x + 3)^2 + 1$ ■

EXAMPLE E Sketch the graph of the function $y = \sin 2x$.

SOLUTION We obtain the graph of $y = \sin 2x$ from that of $y = \sin x$ by compressing horizontally by a factor of 2 (see Figures 7 and 8). Thus, whereas the period of $y = \sin x$ is 2π, the period of $y = \sin 2x$ is $2\pi/2 = \pi$.

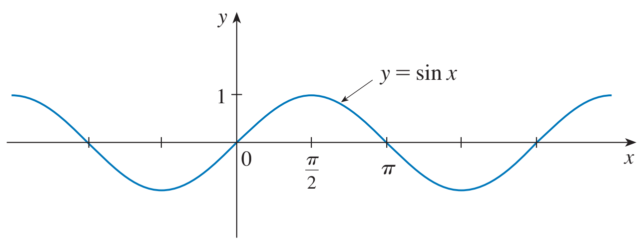


FIGURE 7

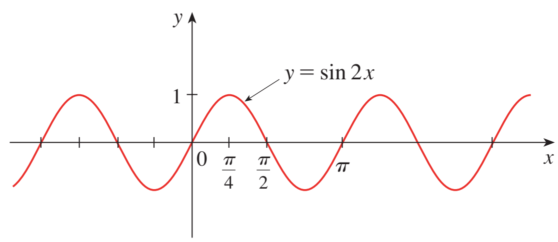


FIGURE 8 ■

EXAMPLE F Figure 9 shows graphs of the number of hours of daylight as functions of the time of the year at several latitudes. Given that Philadelphia is located at approximately 40°N latitude, find a function that models the length of daylight at Philadelphia.

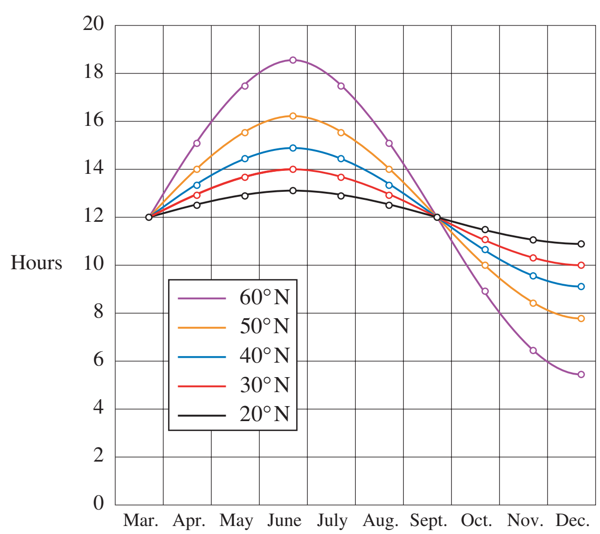


FIGURE 9
Graph of the length of daylight from March 21 through December 21 at various latitudes

Source: Lucia C. Harrison, *Daylight, Twilight, Darkness and Time* (New York: Silver, Burdett, 1935) page 40.

SOLUTION Notice that each curve resembles a shifted and stretched sine function. By looking at the blue curve we see that, at the latitude of Philadelphia, daylight lasts about 14.8 hours on June 21 and 9.2 hours on December 21, so the amplitude of the curve (the factor by which we have to stretch the sine curve vertically) is $\frac{1}{2}(14.8 - 9.2) = 2.8$.

By what factor do we need to stretch the sine curve horizontally if we measure the time t in days? Because there are about 365 days in a year, the period of our model should be 365. But the period of $y = \sin t$ is 2π, so the horizontal stretching factor is $c = 2\pi/365$.

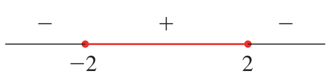
We also notice that the curve begins its cycle on March 21, the 80th day of the year, so we have to shift the curve 80 units to the right. In addition, we shift it 12 units upward. Therefore, we model the length of daylight in Philadelphia on the tth day of the year by the function

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$ ■

EXAMPLE G If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$, find the functions $f + g$, $f - g$, fg, and f/g.

▪ Another way to solve $4 - x^2 \geqslant 0$:

$$(2 - x)(2 + x) \geqslant 0$$



SOLUTION The domain of $f(x) = \sqrt{x}$ is $[0, \infty)$. The domain of $g(x) = \sqrt{4 - x^2}$ consists of all numbers x such that $4 - x^2 \geqslant 0$, that is, $x^2 \leqslant 4$. Taking square roots of both sides, we get $|x| \leqslant 2$, or $-2 \leqslant x \leqslant 2$, so the domain of g is the interval $[-2, 2]$. The intersection of the domains of f and g is

$$[0, \infty) \cap [-2, 2] = [0, 2]$$

Thus, according to the definitions, we have

$$\begin{aligned}
(f + g)(x) &= \sqrt{x} + \sqrt{4 - x^2} && 0 \leqslant x \leqslant 2 \\
(f - g)(x) &= \sqrt{x} - \sqrt{4 - x^2} && 0 \leqslant x \leqslant 2 \\
(fg)(x) &= \sqrt{x}\sqrt{4 - x^2} = \sqrt{4x - x^3} && 0 \leqslant x \leqslant 2 \\
\left(\frac{f}{g}\right)(x) &= \frac{\sqrt{x}}{\sqrt{4 - x^2}} = \sqrt{\frac{x}{4 - x^2}} && 0 \leqslant x < 2
\end{aligned}$$

Notice that the domain of f/g is the interval $[0, 2)$; we have to exclude $x = 2$ because $g(2) = 0$. ■

EXAMPLE H Find $f \circ g \circ h$ if $f(x) = x/(x + 1)$, $g(x) = x^{10}$, and $h(x) = x + 3$.

SOLUTION

$$\begin{aligned}
(f \circ g \circ h)(x) &= f(g(h(x))) = f(g(x + 3)) \\
&= f((x + 3)^{10}) = \frac{(x + 3)^{10}}{(x + 3)^{10} + 1}
\end{aligned}$$ ■