3.3 DERIVATIVES AND THE SHAPES OF GRAPHS

EXAMPLE A Figure 1 shows a population graph for Cyprian honeybees raised in an apiary. How does the rate of population increase change over time? When is this rate highest? Over what intervals is P concave upward or concave downward?

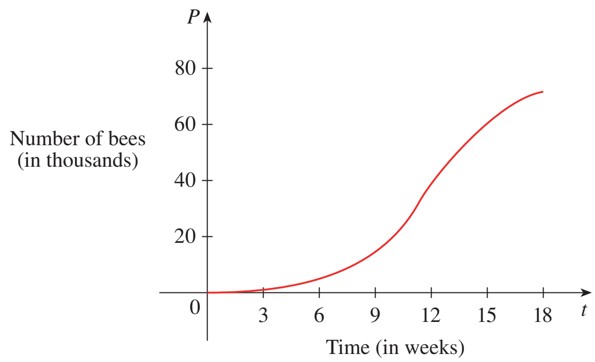

FIGURE 1

SOLUTION By looking at the slope of the curve as t increases, we see that the rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about $t = 12$ weeks, and decreases as the population begins to level off. As the population approaches its maximum value of about 75,000 (called the *carrying capacity*), the rate of increase, $P'(t)$, approaches 0. The curve appears to be concave upward on (0, 12) and concave downward on (12, 18). ■

EXAMPLE B Investigate the family of functions given by $f(x) = cx + \sin x$. What features do the members of this family have in common? How do they differ?

SOLUTION The derivative is $f'(x) = c + \cos x$. If $c > 1$, then $f'(x) > 0$ for all x (since $\cos x \geqslant -1$), so f is always increasing. If $c = 1$, then $f'(x) = 0$ when x is an odd multiple of π, but f just has horizontal tangents there and is still an increasing function. Similarly, if $c \leqslant -1$, then f is always decreasing. If $-1 < c < 1$, then the equation $c + \cos x = 0$ has infinitely many solutions $[x = 2n\pi \pm \cos^{-1}(-c)]$ and f has infinitely many minima and maxima.

The second derivative is $f''(x) = -\sin x$, which is negative when $0 < x < \pi$ and, in general, when $2n\pi < x < (2n + 1)\pi$, where n is any integer. Thus, *all* members of the family are concave downward on $(0, \pi)$, $(2\pi, 3\pi)$, . . . and concave upward on $(\pi, 2\pi)$, $(3\pi, 4\pi)$, ... This is illustrated by several members of the family in Figure 2.

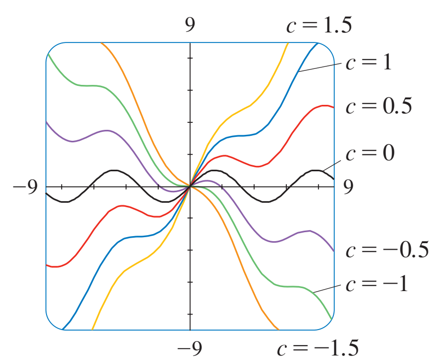

FIGURE 2 ■