ALTERNATIVE TREATMENT:

AREA AND INTEGRALS

SECTION 1 Sigma Notation

In finding areas and evaluating integrals we often encounter sums with many terms. A convenient way of writing such sums uses the Greek letter Σ (capital sigma, corresponding to our letter S) and is called **sigma notation**.

This tells us to end with $i = n$.

This tells us to add. $\displaystyle\sum_{i=m}^{n} a_i$

This tells us to start with $i = m$.

1 Definition If $a_m, a_{m+1}, \ldots, a_n$ are real numbers and m and n are integers such that $m \leqslant n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

With function notation, Definition 1 can be written as

$$\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)$$

Thus, the symbol $\Sigma_{i=m}^{n}$ indicates a summation in which the letter i (called the **index of summation**) takes on the values $m, m+1, \ldots, n$. Other letters can also be used as the index of summation.

EXAMPLE 1

(a) $\displaystyle\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$

(b) $\displaystyle\sum_{i=3}^{n} i = 3 + 4 + 5 + \cdots + (n-1) + n$

(c) $\displaystyle\sum_{j=0}^{5} 2^j = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$

(d) $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

(e) $\displaystyle\sum_{i=1}^{3} \frac{i-1}{i^2+3} = \frac{1-1}{1^2+3} + \frac{2-1}{2^2+3} + \frac{3-1}{3^2+3} = 0 + \frac{1}{7} + \frac{1}{6} = \frac{13}{42}$

(f) $\displaystyle\sum_{i=1}^{4} 2 = 2 + 2 + 2 + 2 = 8$

EXAMPLE 2 Write the sum $2^3 + 3^3 + \cdots + n^3$ in sigma notation.

SOLUTION There is no unique way of writing a sum in sigma notation. We could write

$$2^3 + 3^3 + \cdots + n^3 = \sum_{i=2}^{n} i^3$$

or

$$2^3 + 3^3 + \cdots + n^3 = \sum_{j=1}^{n-1} (j+1)^3$$

or

$$2^3 + 3^3 + \cdots + n^3 = \sum_{k=0}^{n-2} (k+2)^3$$

The following theorem gives three simple rules for working with sigma notation.

2 Theorem If c is any constant (that is, it does not depend on i), then

(a) $\sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i$ (b) $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$

(c) $\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$

Proof To see why these rules are true, all we have to do is write both sides in expanded form. Rule (a) is just the distributive property of real numbers:

$$ca_m + ca_{m+1} + \cdots + ca_n = c(a_m + a_{m+1} + \cdots + a_n)$$

Rule (b) follows from the associative and commutative properties:

$$\begin{aligned}(a_m + b_m) &+ (a_{m+1} + b_{m+1}) + \cdots + (a_n + b_n) \\ &= (a_m + a_{m+1} + \cdots + a_n) + (b_m + b_{m+1} + \cdots + b_n)\end{aligned}$$

Rule (c) is proved similarly.

EXAMPLE 3 Find $\sum_{i=1}^{n} 1$.

SOLUTION

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n$$

||| PRINCIPLE OF MATHEMATICAL INDUCTION

Let S_n be a statement involving the positive integer n. Suppose that

1. S_1 is true.
2. If S_k is true, then S_{k+1} is true.

Then S_n is true for all positive integers n.

EXAMPLE 4 Prove the formula for the sum of the first n positive integers:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

SOLUTION This formula can be proved by mathematical induction (see page 59) or by the following method used by the German mathematician Karl Friedrich Gauss (1777–1855) when he was ten years old.

Write the sum S twice, once in the usual order and once in reverse order:

$$\begin{aligned} S &= 1 + 2 + 3 + \cdots + (n-1) + n \\ S &= n + (n-1) + (n-2) + \cdots + 2 + 1 \end{aligned}$$

Adding all columns vertically, we get

$$2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)$$

On the right side there are n terms, each of which is $n + 1$, so

$$2S = n(n+1) \qquad \text{or} \qquad S = \frac{n(n+1)}{2}$$

EXAMPLE 5 Prove the formula for the sum of the squares of the first n positive integers:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

SOLUTION 1 Let S be the desired sum. We start with the *telescoping sum* (or collapsing sum):

Most terms cancel in pairs.

$$\sum_{i=1}^{n} [(1+i)^3 - i^3] = (2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3) + \cdots + [(n+1)^3 - n^3]$$
$$= (n+1)^3 - 1^3 = n^3 + 3n^2 + 3n$$

On the other hand, using Theorem 2 and Examples 3 and 4, we have

$$\sum_{i=1}^{n} [(1+i)^3 - i^3] = \sum_{i=1}^{n} [3i^2 + 3i + 1] = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$
$$= 3S + 3\frac{n(n+1)}{2} + n = 3S + \tfrac{3}{2}n^2 + \tfrac{5}{2}n$$

Thus we have

$$n^3 + 3n^2 + 3n = 3S + \tfrac{3}{2}n^2 + \tfrac{5}{2}n$$

Solving this equation for S, we obtain

$$3S = n^3 + \tfrac{3}{2}n^2 + \tfrac{1}{2}n$$

or

$$S = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}$$

|||| See pages 59 and 61 for a more thorough discussion of mathematical induction.

SOLUTION 2 Let S_n be the given formula.

1. S_1 is true because $$1^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}$$

2. Assume that S_k is true; that is,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Then

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = (1^2 + 2^2 + 3^2 + \cdots + k^2) + (k+1)^2$$
$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= (k+1)\frac{k(2k+1) + 6(k+1)}{6}$$
$$= (k+1)\frac{2k^2 + 7k + 6}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$
$$= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$$

So S_{k+1} is true.

By the Principle of Mathematical Induction, S_n is true for all n.

We list the results of Examples 3, 4, and 5 together with a similar result for cubes and fourth powers (see Exercises 37–40) as Theorem 3. These formulas are needed for finding areas in the next section.

3 Theorem Let c be a constant and n a positive integer. Then

(a) $\sum_{i=1}^{n} 1 = n$

(b) $\sum_{i=1}^{n} c = nc$

(c) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

(d) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(e) $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

(f) $\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n+1)}{30}$

EXAMPLE 6 Evaluate $\sum_{i=1}^{n} i(4i^2 - 3)$.

SOLUTION Using Theorems 2 and 3, we have

$$\begin{aligned}\sum_{i=1}^{n} i(4i^2 - 3) &= \sum_{i=1}^{n} (4i^3 - 3i) = 4\sum_{i=1}^{n} i^3 - 3\sum_{i=1}^{n} i \\ &= 4\left[\frac{n(n+1)}{2}\right]^2 - 3\,\frac{n(n+1)}{2} \\ &= \frac{n(n+1)[2n(n+1) - 3]}{2} \\ &= \frac{n(n+1)(2n^2+2n-3)}{2}\end{aligned}$$

|||| The type of calculation in Example 7 arises in the next section when we compute areas.

EXAMPLE 7 Find $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right]$.

SOLUTION

$$\begin{aligned}\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(\frac{i}{n}\right)^2 + 1\right] &= \lim_{n\to\infty} \sum_{i=1}^{n}\left[\frac{3}{n^3}i^2 + \frac{3}{n}\right] \\ &= \lim_{n\to\infty}\left[\frac{3}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n} 1\right] \\ &= \lim_{n\to\infty}\left[\frac{3}{n^3}\,\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\cdot n\right] \\ &= \lim_{n\to\infty}\left[\frac{1}{2}\cdot\frac{n}{n}\cdot\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) + 3\right] \\ &= \lim_{n\to\infty}\left[\frac{1}{2}\cdot 1\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 3\right] \\ &= \tfrac{1}{2}\cdot 1\cdot 1\cdot 2 + 3 = 4\end{aligned}$$

1 Exercises

1–10 Write the sum in expanded form.

1. $\sum_{i=1}^{5} \sqrt{i}$

2. $\sum_{i=1}^{6} \frac{1}{i+1}$

3. $\sum_{i=4}^{6} 3^i$

4. $\sum_{i=4}^{6} i^3$

5. $\sum_{k=0}^{4} \frac{2k-1}{2k+1}$

6. $\sum_{k=5}^{8} x^k$

7. $\sum_{i=1}^{n} i^{10}$

8. $\sum_{j=n}^{n+3} j^2$

9. $\sum_{j=0}^{n-1} (-1)^j$

10. $\sum_{i=1}^{n} f(x_i)\,\Delta x_i$

11–20 Write the sum in sigma notation.

11. $1 + 2 + 3 + 4 + \cdots + 10$

12. $\sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7}$

13. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots + \frac{19}{20}$

14. $\frac{3}{7} + \frac{4}{8} + \frac{5}{9} + \frac{6}{10} + \cdots + \frac{23}{27}$

15. $2 + 4 + 6 + 8 + \cdots + 2n$

16. $1 + 3 + 5 + 7 + \cdots + (2n-1)$

17. $1 + 2 + 4 + 8 + 16 + 32$

18. $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}$

19. $x + x^2 + x^3 + \cdots + x^n$

20. $1 - x + x^2 - x^3 + \cdots + (-1)^n x^n$

21–36 Find the value of the sum.

21. $\sum_{i=4}^{8} (3i - 2)$

22. $\sum_{i=3}^{6} i(i+2)$

23. $\sum_{j=1}^{6} 3^{j+1}$

24. $\sum_{k=0}^{8} \cos k\pi$

25. $\sum_{n=1}^{20} (-1)^n$

26. $\sum_{i=1}^{100} 4$

27. $\sum_{i=0}^{4} (2^i + i^2)$

28. $\sum_{i=-2}^{4} 2^{3-i}$

29. $\sum_{i=1}^{n} 2i$

30. $\sum_{i=1}^{n} (2 - 5i)$

31. $\sum_{i=1}^{n} (i^2 + 3i + 4)$

32. $\sum_{i=1}^{n} (3 + 2i)^2$

33. $\sum_{i=1}^{n} (i+1)(i+2)$

34. $\sum_{i=1}^{n} i(i+1)(i+2)$

35. $\sum_{i=1}^{n} (i^3 - i - 2)$

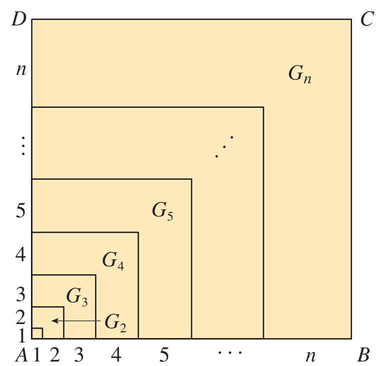
36. $\sum_{i=1}^{n} k^2(k^2 - k + 1)$

37. Prove formula (b) of Theorem 3.

38. Prove formula (e) of Theorem 3 using mathematical induction.

39. Prove formula (e) of Theorem 3 using a method similar to that of Example 5, Solution 1 [start with $(1+i)^4 - i^4$].

40. Prove formula (e) of Theorem 3 using the following method published by Abu Bekr Mohammed ibn Alhusain Alkarchi in about A.D. 1010. The figure shows a square $ABCD$ in which sides AB and AD have been divided into segments of lengths 1, 2, 3, . . . , n. Thus the side of the square has length $n(n+1)/2$ so the area is $[n(n+1)/2]^2$. But the area is also the sum of the areas of the n "gnomons" $G_1, G_2, \ldots, G_n$ shown in the figure. Show that the area of G_i is i^3 and conclude that formula (e) is true.



41. Evaluate each telescoping sum.

(a) $\sum_{i=1}^{n} [i^4 - (i-1)^4]$

(b) $\sum_{i=1}^{100} (5^i - 5^{i-1})$

(c) $\sum_{i=3}^{99} \left(\frac{1}{i} - \frac{1}{i+1}\right)$

(d) $\sum_{i=1}^{n} (a_i - a_{i-1})$

42. Prove the generalized triangle inequality

$$\left|\sum_{i=1}^{n} a_i\right| \leqslant \sum_{i=1}^{n} |a_i|$$

43–46 Find each limit.

43. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2$

44. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^3 + 1\right]$

45. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{2i}{n}\right)^3 + 5\left(\frac{2i}{n}\right)\right]$

46. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(1 + \frac{3i}{n}\right)^3 - 2\left(1 + \frac{3i}{n}\right)\right]$

47. Prove the formula for the sum of a finite geometric series with first term a and common ratio $r \neq 1$:

$$\sum_{i=1}^{n} ar^{i-1} = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

48. Evaluate $\sum_{i=1}^{n} \frac{3}{2^{i-1}}$.

49. Evaluate $\sum_{i=1}^{n} (2i + 2^i)$.

50. Evaluate $\sum_{i=1}^{m} \left[\sum_{j=1}^{n} (i + j) \right]$

51. Find the number n such that $\sum_{i=1}^{n} i = 78$.

52. (a) Use the product formula for $\sin x \cos y$ (see 18a in Appendix D) to show that

$$2 \sin \tfrac{1}{2}x \cos ix = \sin\left(i + \tfrac{1}{2}\right)x - \sin\left(i - \tfrac{1}{2}\right)x$$

(b) Use the identity in part (a) and telescoping sums to prove the formula

$$\sum_{i=1}^{n} \cos ix = \frac{\sin\left(n + \frac{1}{2}\right)x - \sin \frac{1}{2}x}{2 \sin \frac{1}{2}x}$$

where x is not an integer multiple of 2π. Deduce that

$$\sum_{i=1}^{n} \cos ix = \frac{\sin \frac{1}{2}nx \cos \frac{1}{2}(n + 1)x}{\sin \frac{1}{2}x}$$

53. Use the method of Exercise 52 to prove the formula

$$\sum_{i=1}^{n} \sin ix = \frac{\sin \frac{1}{2}nx \sin \frac{1}{2}(n + 1)x}{\sin \frac{1}{2}x}$$

where x is not an integer multiple of 2π.

SECTION 2 Area

We begin by attempting to solve the *area problem:* Find the area of the region S that lies under the curve $y = f(x)$ from a to b. This means that S, illustrated in Figure 1, is bounded by the graph of a continuous function f [where $f(x) \geqslant 0$], the vertical lines $x = a$ and $x = b$, and the x-axis.

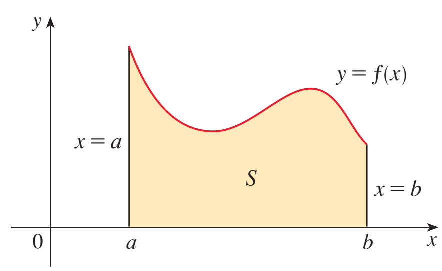


FIGURE 1
$S = \{(x, y) \mid a \leqslant x \leqslant b, 0 \leqslant y \leqslant f(x)\}$

In trying to solve the area problem we have to ask ourselves: What is the meaning of the word *area*? This question is easy to answer for regions with straight sides. For a rectangle, the area is defined as the product of the length and the width. The area of a triangle is half the base times the height. The area of a polygon is found by dividing it into triangles (as in Figure 2) and adding the areas of the triangles.

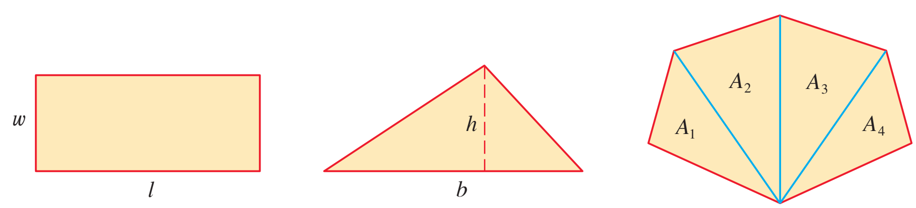


FIGURE 2 $A = lw$ $\quad A = \frac{1}{2}bh \quad A = A_1 + A_2 + A_3 + A_4$

However, it isn't so easy to find the area of a region with curved sides. We all have an intuitive idea of what the area of a region is. But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations. We pursue a similar idea for areas. We first approximate the region S by polygons and then we take the limit of the areas of these polygons. The following example illustrates the procedure.

EXAMPLE 1 Let's try to find the area under the parabola $y = x^2$ from 0 to 1 (the parabolic region S illustrated in Figure 3).

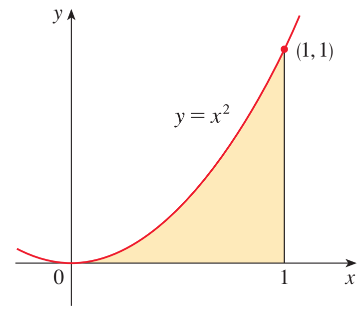


FIGURE 3

One method of approximating the desired area is to divide the interval $[0, 1]$ into subintervals of equal length and consider the rectangles whose bases are these subintervals and whose heights are the values of the function at the right-hand endpoints of these subintervals. Figure 4 shows the approximation of the parabolic region by four, eight, and n rectangles

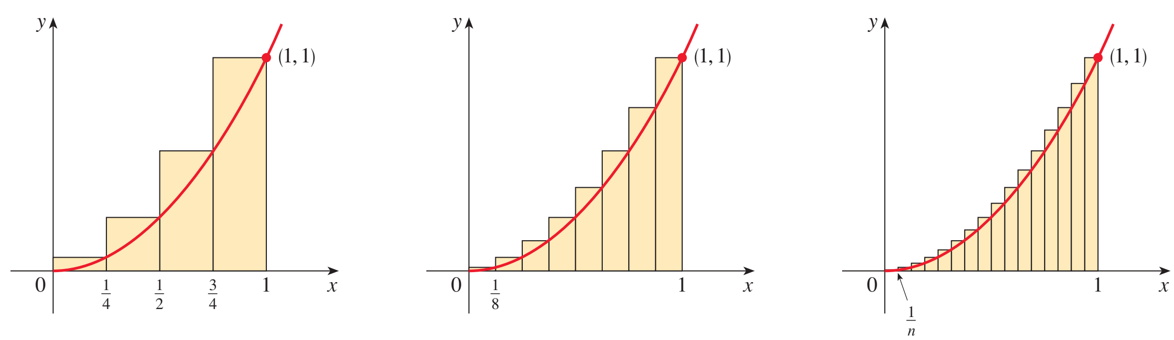


FIGURE 4

Let S_n be the sum of the areas of the n rectangles in Figure 4(c). Each rectangle has width $1/n$ and the heights are the values of the function $f(x) = x^2$ at the points $1/n, 2/n, 3/n, \dots, n/n$; that is, the heights are $(1/n)^2, (2/n)^2, (3/n)^2, \dots, (n/n)^2$. Thus

$$\begin{aligned} S_n &= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2 \\ &= \frac{1}{n}\,\frac{1}{n^2}(1^2 + 2^2 + 3^2 + \cdots + n^2) \\ &= \frac{1}{n^3}\sum_{i=1}^{n} i^2 \end{aligned}$$

Using the formula for the sum of the squares of the first n integers [Formula 1.3(d)], we can write

$$S_n = \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

For instance, the sum of the areas of the four shaded rectangles in Figure 4(a) is

$$S_4 = \frac{5(9)}{6(16)} = 0.46875$$

n	S_n
10	0.385000
20	0.358750
30	0.350185
50	0.343400
100	0.338350
1000	0.333834

and the sum of the areas of the eight rectangles in FIgure 4(b) is

$$S_8 = \frac{9(17)}{6(64)} = 0.3984375$$

The results of similar calculations are shown in the table in the margin.

It looks as if S_n is becoming closer to $\frac{1}{3}$ as n increases. In fact

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \\ &= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \\ &= \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \\ &= \tfrac{1}{6}\cdot 1\cdot 2 = \tfrac{1}{3} \end{aligned}$$

From Figure 4 it appears that, as n increases, S_n becomes a better and better approximation to the area of the parabolic segment. Therefore we *define* the area A to be the limit of the sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n\to\infty} S_n = \tfrac{1}{3}$$

In applying the idea of Example 1 to the more general region S of Figure 1, we have no need to use rectangles of equal width. We start by subdividing the interval $[a, b]$ into n smaller subintervals by choosing partition points $x_0, x_1, x_2, \ldots, x_n$ so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Then the n subintervals are

$$[x_0, x_1], \quad [x_1, x_2], \quad [x_2, x_3], \quad \ldots, \quad [x_{n-1}, x_n]$$

This subdivision is called the **partition** of $[a, b]$ and we denote it by P. We use the notation Δx_i for the length of the ith subinterval $[x_{i-1}, x_i]$. Thus

$$\Delta x_i = x_i - x_{i-1}$$

This length of the longest subinterval is denoted by $\|P\|$ and is called the **norm** of P. Thus

$$\|P\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$$

Figure 5 illustrates one possible partition of $[a, b]$.

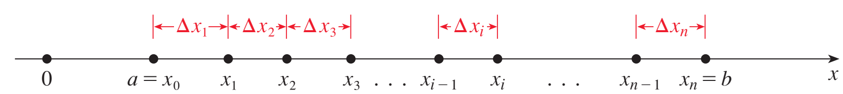


FIGURE 5

By drawing the lines $x = a, x = x_1, x = x_2, \ldots, x = b$, we use the partition P to divide the region S into strips $S_1, S_2, \ldots, S_n$ as in Figure 6. Next we approximate these strips S_i by rectangles R_i. To do this we choose a number x_i^* in each subinterval $[x_{i-1}, x_i]$ and construct a rectangle R_i with base Δx_i and height $f(x_i^*)$ as in Figure 7.

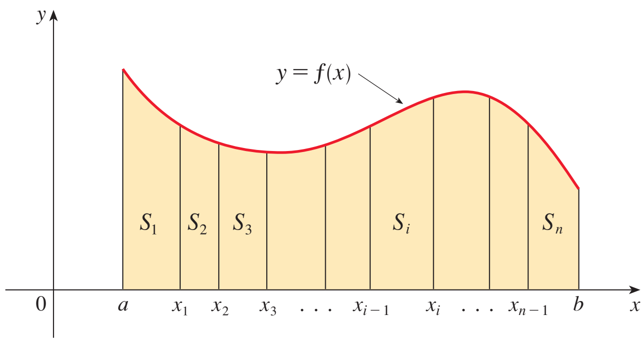


FIGURE 6

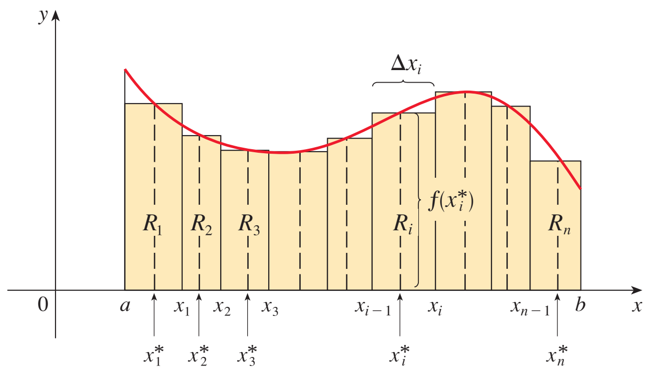


FIGURE 7

Each point x_i^* can be anywhere in its subinterval—at the right endpoint (as in Example 1) or at the left endpoint or somewhere between the endpoints. The area of the ith rectangle R_i is

$$A_i = f(x_i^*)\,\Delta x_i$$

The n rectangles $R_1, \ldots, R_n$ form a polygonal approximation to the region S. What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$\boxed{1} \qquad \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = f(x_1^*)\,\Delta x_1 + \cdots + f(x_n^*)\,\Delta x_n$$

Figure 8 shows this approximation for partitions with $n = 2$, 4, 8, and 12.

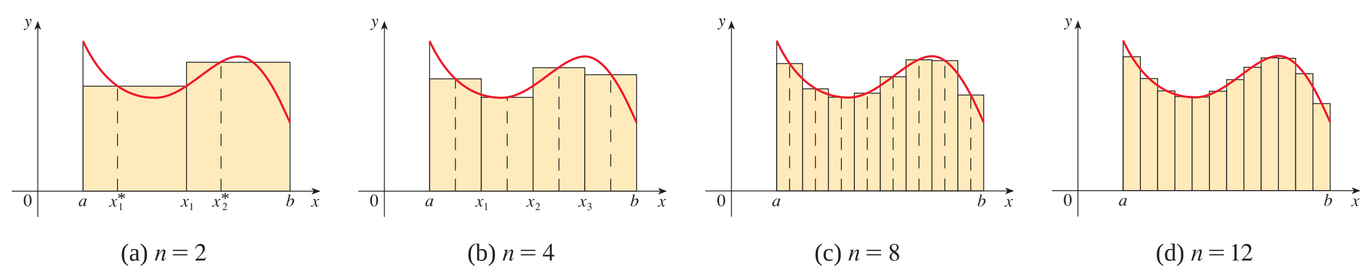


FIGURE 8

Notice that this approximation appears to become better and better as the strips become thinner and thinner, that is, as $\|P\| \to 0$. Therefore we define the **area** A of the region S as the limiting value (if it exists) of the areas of the approximating polygons, that is, the limit of the sum (1) of the areas of the approximating rectangles. In symbols:

$$\boxed{2} \qquad A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

The preceding discussion and the diagrams in Figures 7 and 8 show that the definition of area in (2) corresponds to our intuitive feeling of what area ought to be.

The limit in (2) may or may not exist. It can be shown that if f is continuous, then this limit does exist; that is, the region has an area. [The precise meaning of the limit in Definition 2 is that for every $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\left| A - \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i \right| < \varepsilon \qquad \text{whenever} \qquad \|P\| < \delta$$

In other words, the area can be approximated by a sum of areas of rectangles to within an arbitrary degree of accuracy (ε) by taking the norm of the partition sufficiently small.

EXAMPLE 2
(a) If the interval $[0, 3]$ is divided into subintervals by the partition P and the set of partition points is $\{0, 0.6, 1.2, 1.6, 2.5, 3\}$, find $\|P\|$.
(b) If $f(x) = x^2 - 4x + 5$ and x_i^* is chosen to be the left endpoint of the ith subinterval, find the sum of the areas of the approximating rectangles.
(c) Sketch the approximating rectangles.

SOLUTION
(a) We are given that $x_0 = 0$, $x_1 = 6$, $x_2 = 1.2$, $x_3 = 1.6$, $x_4 = 2$, $x_5 = 2.5$, and $x_6 = 3$, so

$$\Delta x_1 = 0.6 - 0 = 0.6 \qquad \Delta x_2 = 1.2 - 0.6 = 0.6$$

$$\Delta x_3 = 1.6 - 1.2 = 0.4 \qquad \Delta x_4 = 2 - 1.6 = 0.4$$

$$\Delta x_5 = 2.5 - 2 = 0.5 \qquad \Delta x_6 = 3 - 2.5 = 0.5$$

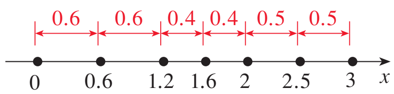


FIGURE 9

(See Figure 9.) Therefore

$$\|P\| = \max\{0.6, 0.6, 0.4, 0.4, 0.5, 0.5\} = 6$$

(b) Since $x_i^* = x_{i-1}$, the sum of the areas of the approximating rectangles is, by (1),

$$\sum_{i=1}^{6} f(x_i^*)\,\Delta x_i = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x_i$$

$$= f(0)\,\Delta x_1 + f(0.6)\,\Delta x_2 + f(1.2)\,\Delta x_3 + f(1.6)\,\Delta x_4 + f(2)\,\Delta x_5 + f(2.5)\,\Delta x_6$$

$$= 5(0.6) + 2.96(0.6) + 1.64(0.4) + 1.16(0.4) + 1(0.5) + 1.25(0.5)$$

$$= 7.021$$

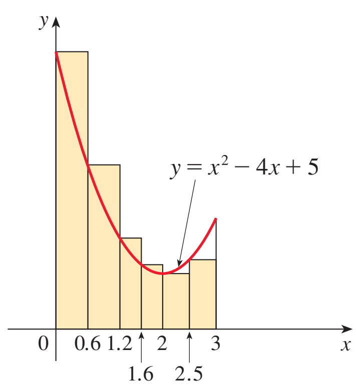


FIGURE 10

(c) The graph of f and the approximating rectangles are sketched in Figure 10.

EXAMPLE 3 Find the area under the parabola $y = x^2 + 1$ from 0 to 2.

SOLUTION Since $f(x) = x^2 + 1$ is continuous, the limit (2) that defines the area must exist for all possible partitions P of the interval $[0, 2]$ as long as $\|P\| \to 0$. To simplify things let us take the partition P that divides $[0, 2]$ into n subintervals of equal length. (This is called a regular partition.) Then the partition points are

$$x_0 = 0,\ x_1 = \frac{2}{n},\ x_2 = \frac{4}{n},\ \ldots,\ x_i = \frac{2i}{n},\ \ldots,\ x_n = \frac{2n}{n} = 2$$

and

$$\Delta x_1 = \Delta x_2 = \cdots = \Delta x_i = \cdots = \Delta x_n = \frac{2}{n}$$

so the norm of P is

$$\|P\| = \max\{\Delta x_i\} = \frac{2}{n}$$

The point x_i^* can be chosen to be anywhere in the ith subinterval. For the sake of definiteness, let us choose it to be the right-hand endpoint:

$$x_i^* = x_i = \frac{2i}{n}$$

Since $\|P\| = 2/n$, the condition $\|P\| \to 0$ is equivalent to $n \to \infty$. So the definition of area (2) becomes

$$A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{2i}{n}\right)^2 + 1\right]\frac{2}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{8i^2}{n^3} + \frac{2}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} 1\right] \qquad \text{(by Theorem 1.2)}$$

$$= \lim_{n\to\infty} \left[\frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{2}{n} \cdot n\right] \qquad \text{(by Theorem 1.3)}$$

$$= \lim_{n\to\infty} \left[\frac{4}{3} \cdot 1 \cdot \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 2\right]$$

$$= \tfrac{4}{3} \cdot 1 \cdot 1 \cdot 2 + 2 = \tfrac{14}{3}$$

The sum in this calculation is represented by the areas of the shaded rectangles in Figure 11. Notice that in this case, with our choice of x_i^* as the right-hand endpoint and

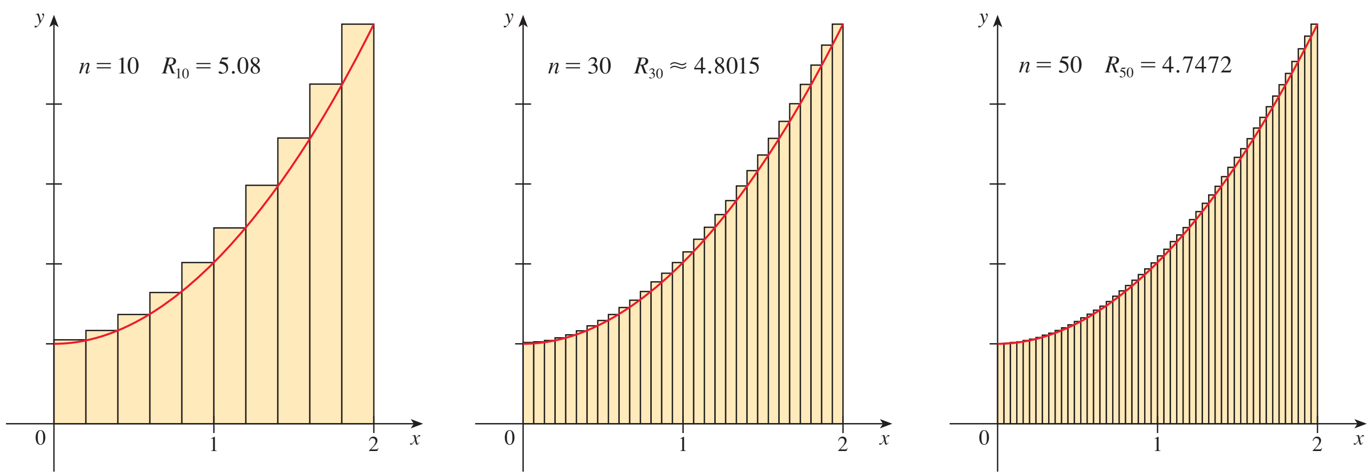


FIGURE 11 Right sums

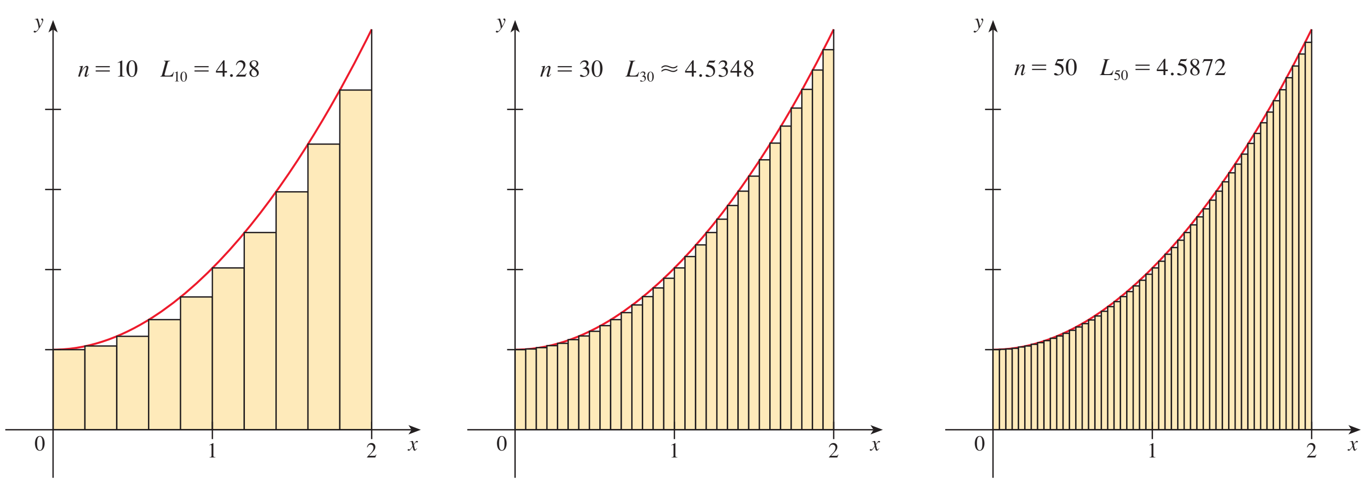


FIGURE 12 Left sums

since f is increasing, $f(x_i^*)$ is the maximum value of f on $[x_{i-1}, x_i]$, so the sum R_n of the areas of the approximating rectangles is always *greater* than the exact area $A = \frac{14}{3}$.

We could just as well have chosen x_i^* to be the left-hand endpoint, that is, $x_i^* = x_{i-1} = 2(i-1)/n$. Then $f(x_i^*)$ is the minimum value of f on $[x_{i-1}, x_i]$, so the sum L_n of the areas of the approximating rectangles in Figure 12 is always less than A.

The calculation with this choice is as follows:

$$\begin{aligned}
A &= \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2(i-1)}{n}\right)\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left\{\left[\frac{2(i-1)}{n}\right]^2 + 1\right\}\frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{8}{n^3}(i^2 - 2i + 1) + \frac{2}{n}\right] \\
&= \lim_{n\to\infty} \left[\frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{16}{n^3}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} 1 + \frac{2}{n}\sum_{i=1}^{n} 1\right] \\
&= \lim_{n\to\infty} \left[\frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6} - \frac{16}{n^3}\,\frac{n(n+1)}{2} + \frac{8}{n^3}n + \frac{2}{n}n\right] \\
&= \lim_{n\to\infty} \left[\tfrac{4}{3}\cdot 1\cdot\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - \frac{8}{n}\left(1 + \frac{1}{n}\right) + \frac{8}{n^2} + 2\right] \\
&= \tfrac{4}{3}\cdot 1\cdot 1\cdot 2 - 0\cdot 1 + 0 + 2 = \tfrac{14}{3}
\end{aligned}$$

Notice that we have obtained the same answer with the different choice of x_i^*. In fact, we would obtain the same answer if x_i^* was chosen to be the midpoint of $[x_{i-1}, x_i]$ (see Exercise 11) or indeed any other point of this interval.

EXAMPLE 4 Find the area under the cosine curve from 0 to b, where $0 \leqslant b \leqslant \pi/2$.

SOLUTION As in the first part of Example 3, we choose a regular partition P so that

$$\|P\| = \Delta x_1 = \Delta x_2 = \cdots = \Delta x_n = \frac{b}{n}$$

and we choose x_i^* to be the right-hand endpoint of the ith subinterval:

$$x_i^* = x_i = \frac{ib}{n}$$

Since $\|P\| = b/n \to 0$ as $n \to \infty$, the area under the cosine curve from 0 to b is

3 $$\begin{aligned}
A &= \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(i\frac{b}{n}\right)\frac{b}{n} \\
&= \lim_{n\to\infty} \frac{b}{n}\sum_{i=1}^{n} \cos\left(i\frac{b}{n}\right)
\end{aligned}$$

To evaluate this limit we use the formula of Exercise 52 in Section 1:

$$\sum_{i=1}^{n} \cos ix = \frac{\sin\frac{1}{2}nx\cos\frac{1}{2}(n+1)x}{\sin\frac{1}{2}x}$$

with $x = b/n$. Then Equation 3 becomes

4 $$A = \lim_{n \to \infty} \frac{b}{n} \frac{\sin \frac{1}{2} b \cos \left[\dfrac{(n+1)b}{2n} \right]}{\sin \dfrac{b}{2n}}$$

Now $$\cos \left[\frac{(n+1)b}{2n} \right] = \cos \left(1 + \frac{1}{n} \right) \frac{b}{2} \to \cos \frac{b}{2} \quad \text{as } n \to \infty$$

since cosine is continuous. Letting $t = b/n$ and using Theorem 3.5.2, we have

$$\lim_{n \to \infty} \frac{b}{n} \cdot \frac{1}{\sin \dfrac{b}{2n}} = \lim_{t \to 0^+} \frac{t}{\sin \dfrac{t}{2}} = \lim_{t \to 0^+} 2 \cdot \frac{\dfrac{t}{2}}{\sin \dfrac{t}{2}} = 2$$

Putting these limits in Equation 4, we obtain

$$A = 2 \sin \frac{b}{2} \cos \frac{b}{2} = \sin b$$

In particular, taking $b = \pi/2$, we have proved that the area under the cosine curve from 0 to $\pi/2$ is $\sin(\pi/2) = 1$ (see Figure 13).

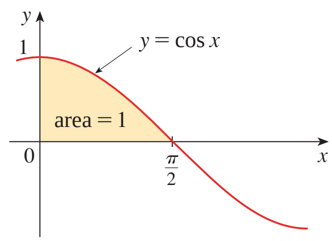


FIGURE 13

NOTE ◦ The area calculations in Example 3 and 4 are not easy. We will see in Section 5.3, however, that the Fundamental Theorem of Calculus gives a much easier method for computing these areas.

2 Exercises

1–8 You are given a function f, an interval, partition points, and a description of x_i^* within the ith subinterval.

(a) Find $\|P\|$.

(b) Find the sum of the areas of the approximating rectangles, as given in (1).

(c) Sketch the graph of f and the approximating rectangles.

1. $f(x) = 16 - x^2$, $[0, 4]$, $\{0, 1, 2, 3, 4\}$, $x_i^* =$ left endpoint

2. $f(x) = 16 - x^2$, $[0, 4]$, $\{0, 1, 2, 3, 4\}$, $x_i^* =$ right endpoint

3. $f(x) = 16 - x^2$, $[0, 4]$, $\{0, 1, 2, 3, 4\}$, $x_i^* =$ midpoint

4. $f(x) = 2x + 1$, $[0, 4]$, $\{0, 0.5, 1, 2, 4\}$, $x_i^* =$ left endpoint

5. $f(x) = x^3 + 2$, $[-1, 2]$, $\{-1, -0.5, 0, 0.5, 1.0, 1.5, 2\}$, $x_i^* =$ right endpoint

6. $f(x) = 1/(x + 1)$, $[0, 2]$, $\{0, 0.5, 1.0, 1.5, 2\}$, $x_1^* = 0.25$, $x_2^* = 1$, $x_3^* = 1.25$, $x_4^* = 2$

7. $f(x) = 2 \sin x$, $[0, \pi]$, $\{0, \pi/4, \pi/2, 3\pi/4, \pi\}$, $x_1^* = \pi/6$, $x_2^* = \pi/3$, $x_3^* = 2\pi/3$, $x_4^* = 5\pi/6$

8. $f(x) = 4 \cos x$, $[0, \pi/2]$, $\{0, \pi/6, \pi/4, \pi/3, \pi/2\}$, $x_i^* =$ left endpoint

9. (a) Sketch a graph of the region that lies under the parabola $y = x^2 - 2x + 2$ from $x = 0$ to $x = 3$ and use it to make a rough visual estimate of the area of the region.
(b) Find an expression for R_n, the sum of the areas of the n approximating rectangles, taking x_i^* in (1) to be the right endpoint and using subintervals of equal length.
(c) Find the numerical values of the approximating areas R_n for $n = 6$, 12, and 24.
(d) Find the exact area of the region.

10. (a) Use a graphing device to sketch a graph of the region that lies under the curve $y = 4x - x^3$ from $x = 0$ to $x = 2$ and use it to make a rough visual estimate of the area of the region.
(b) Find an expression for R_n, the sum of the areas of the n approximating rectangles, taking x_i^* in (1) to be the right endpoint and using subintervals of equal length.
(c) Find the numerical values of the approximating areas R_n for $n = 10$, 20, and 30.
(d) Find the exact area of the region.

11. Find the area from Example 3 taking x_i^* to be the midpoint of $[x_{i-1}, x_i]$. Illustrate the approximating rectangles with a sketch.

12. Find the area under the curve $y = x^3$ from 0 to 1 using subintervals of equal length and taking x_i^* in (2) to be the (a) left endpoint, (b) right endpoint, and (c) midpoint of the ith subinterval. In each case, sketch the approximating rectangles.

13–18 Use (2) to find the area under the given curve from a to b. Use equal subintervals and take x_i^* to be the right endpoint of the ith subinterval. Sketch the region.

13. $y = 2x + 1, \quad a = 0, b = 5$

14. $y = x^2 + 3x - 2, \quad a = 1, b = 4$

15. $y = 2x^2 - 4x + 5, \quad a = -3, b = 2$

16. $y = x^3 + 2x, \quad a = 0, b = 2$

17. $y = x^3 + 2x^2 + x, \quad a = 0, b = 1$

18. $y = x^4 + 3x + 2, \quad a = 0, b = 3$

19–20 If you have a programmable calculator (or a computer), it is possible to evaluate the expression (1) for the sum of areas of approximating rectangles, even for large values of n, using looping. (On a TI use the Is> command, on a Casio use Isz, on an HP or in BASIC use a FOR-NEXT loop.) Compute the sum of the areas of approximating rectangles using equal subintervals and right endpoints for $n = 10$, 30, and 50. Then guess the value of the exact area.

19. The region under $y = \sin x$ from 0 to π

20. The region under $y = 1/x^2$ from 1 to 2

CAS **21.** Some computer algebra systems have commands that will draw approximating rectangles and evaluate the sums of their areas, at least if x_i^* is a left or right endpoint. (For instance, in Maple use leftbox, rightbox, leftsum, and rightsum.)
(a) If $f(x) = \sqrt{x}$, $1 \leqslant x \leqslant 4$, find the left and right sums for $n = 10$, 30, and 50.
(b) Illustrate by graphing the rectangles in part (a).
(c) Show that the exact area under f lies between 4.6 and 4.7.

CAS **22.** (a) If $f(x) = \sin(\sin x)$, $0 \leqslant x \leqslant \pi/2$, use the commands discussed in Exercise 21 to find the left and right sums for $n = 10$, 30, and 50.
(b) Illustrate by graphing the rectangles in part (a).
(c) Show that the exact area under f lies between 0.87 and 0.91.

23–24 Determine a region whose area is equal to the given limit. Do not evaluate the limit.

23. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n}$

24. $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} \sqrt{1 + \frac{3i}{n}}$

25. Find the area under the curve $y = \sin x$ from 0 to π. [*Hint:* Use equal subintervals and right endpoints, and use Exercise 53 in Section 1.]

26. (a) Let A_n be the area of a polygon with n equal sides inscribed in a circle with radius r. By dividing the polygon into n congruent triangles with central angle $2\pi/n$, show that $A_n = \frac{1}{2} n r^2 \sin(2\pi/n)$.
(b) Show that $\lim_{n\to\infty} A_n = \pi r^2$. [*Hint:* Use Equation 3.5.2.]

SECTION 3 The Definite Integral

We saw in the preceeding section that a limit of the form

1 $$A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

arises when we compute an area. It turns out that this same type of limit occurs in a wide variety of situations even when f is not necessarily a positive function. In Chapters 5 and 8 we will see that limits of the form (1) also arise in finding lengths of curves, volumes of solids, areas of surfaces, centers of mass, fluid pressure, and work, as well as other quantities. We therefore give this type of limit a special name and notation.

2 Definition of a Definite Integral If f is a function defined on a closed interval $[a, b]$, let P be a partition of $[a, b]$ with partition points $x_0, x_1, \ldots, x_n$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Choose points x_i^* in $[x_{i-1}, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$ and $\|P\| = \max\{\Delta x_i\}$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

if this limit exists. If the limit does exist, then f is called **integrable** on the interval $[a, b]$.

NOTE 1 ▫ The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and was chosen because an integral is a limit of sums. In the notation $\int_a^b f(x)\,dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**. The symbol dx has no meaning by itself; $\int_a^b f(x)\,dx$ is all one symbol. The procedure of calculating an integral is called **integration**.

NOTE 2 ▫ The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on x. In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

NOTE 3 ▫ The sum

3 $$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

that occurs in Definition 2 is called a **Riemann sum** after the German mathematician Bernhard Riemann (1826–1866). The definite integral is sometimes called the **Riemann integral**. If f happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles [Compare (3) with (2.1).] If f takes on both positive and negative values, as in Figure 1, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the *negatives* of the areas of the rectangles that lie below the x-axis (the areas of the gold rectangles *minus* the areas of the blue rectangles).

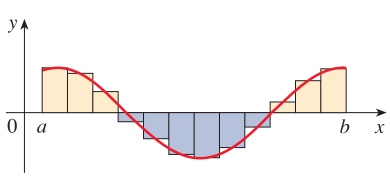


FIGURE 1

EXAMPLE 1 Let $f(x) = 1 + 5x$ and consider the partition P of the interval $[-2, 1]$ by means of the set of partition points $\{-2, -1.5, -1, -0.3, 0.2, 1\}$. In this example

$a = -2$, $b = 1$, $n = 5$, and $x_0 = -2$, $x_1 = -1.5$, $x_2 = -1$, $x_3 = -0.3$, $x_4 = 0.2$, and $x_5 = 1$. the lengths of the subintervals are

$$\Delta x_1 = -1.5 - (-2) = 0.5 \qquad \Delta x_2 = -1 - (-1.5) = 0.5$$

$$\Delta x_3 = -0.3 - (-1) = 0.7 \qquad \Delta x_4 = 0.2 - (-0.3) = 0.5$$

$$\Delta x_5 = 1 - 0.2 = 0.8$$

Thus the norm of the partition P is

$$\|P\| = \max\{0.5, 0.5, 0.7, 0.5, 0.8\} = 0.8$$

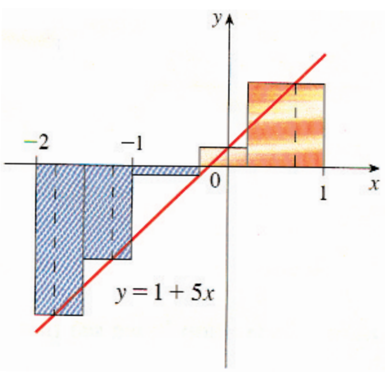


FIGURE 2

Suppose we choose $x_1^* = -1.8$, $x_2^* = -1.2$, $x_3^* = -0.3$, $x_4^* = 0$, and $x_5^* = 0.7$. Then the corresponding Riemann sum is

$$\begin{aligned}\sum_{i=1}^{5} f(x_i^*)\,\Delta x_i &= f(-1.8)\Delta x_1 + f(-1.2)\Delta x_2 + f(-0.3)\Delta x_3 + f(0)\Delta x_4 + f(0.7)\Delta x_5 \\ &= (-8)(0.5) + (-5)(0.5) + (-0.5)(0.7) + 1(0.5) + (4.5)(0.8) \\ &= -2.75\end{aligned}$$

Notice that, in this example, f is not a positive function and so the Riemann sum does not represent a sum of areas of rectangles. But it does represent the sum of the areas of the gold rectangles (above the x-axis) minus the sum of the areas of the blue rectangles (below the axis) in Figure 2.

NOTE 4 ▫ An integral need not represent an area. But for *positive* functions, an integral can be interpreted as an area. In fact, comparing Definition 2 with the definition of area (2.2), we see the following:

> For the special case where $f(x) \geqslant 0$,
>
> $$\int_a^b f(x)\,dx = \text{the area under the graph of } f \text{ from } a \text{ to } b$$

In general, a definite integral can be interpreted as a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

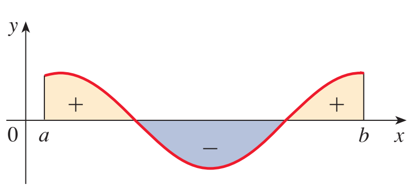


FIGURE 3

where A_1 is the area of the region above the x-axis and below the graph of f and A_2 is the area of the region below the x-axis and above the graph of f. (This seems reasonable from a comparison of Figures 1 and 3.)

NOTE 5 ▫ The precise meaning of the limit that defines the integral in Definition 2 is as follows:

$\int_a^b f(x)\,dx = I$ means that for every $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\left| I - \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i \right| < \varepsilon$$

for all partitions P of $[a, b]$ with $\|P\| < \delta$ and for all possible choices of x_i^* in $[x_{i-1}, x_i]$.

|||| Bernhard Riemann received his Ph.D. under the direction of the legendary Gauss at the University of Göttingen and remained there to teach. Gauss, who was not in the habit of praising other mathematicians, spoke of Riemann's "creative, active, truly mathematical mind and gloriously fertile originality." The definition (2) of an integral that we use is due to Riemann. He also made major contributions to the theory of functions of a complex variable, mathematical physics, number theory, and the foundations of geometry. Riemann's broad concept of space and geometry turned out to be the right setting, 50 years later, for Einstein's general relativity theory. Riemann's health was poor throughout his life, and he died of tuberculosis at the age of 39.

This means that a definite integral can be approximated to within any desired degree of accuracy by a Riemann sum.

NOTE 6 ▫ In Definition 2 we are dealing with a function f defined on an interval $[a, b]$, so we are implicitly assuming that $a < b$. But for some purposes it is useful to extend the definition of $\int_a^b f(x)\,dx$ to the case where $a > b$ or $a = b$ as follows:

If $a > b$, then $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

If $a = b$, then $\displaystyle\int_a^a f(x)\,dx = 0$.

EXAMPLE 2 Express

$$\lim_{\|P\| \to 0} \sum_{i=1}^{n} [(x_i^*)^3 + x_i^* \sin x_i^*]\,\Delta x_i$$

as an integral on the interval $[0, \pi]$.

SOLUTION Comparing the given limit with the limit in Definition 2, we see that they will be identical if we choose

$$f(x) = x^3 + x \sin x$$

We are given that $a = 0$ and $b = \pi$. Therefore, by Definition 2, we have

$$\lim_{\|P\| \to 0} \sum_{i=1}^{n} [(x_i^*)^3 + x_i^* \sin x_i^*]\,\Delta x_i = \int_0^\pi (x^3 + x \sin x)\,dx$$

EXAMPLE 3 Evaluate the following integrals by interpreting each in terms of areas.

(a) $\displaystyle\int_0^1 \sqrt{1 - x^2}\,dx$ (b) $\displaystyle\int_0^3 (x - 1)\,dx$

SOLUTION

(a) Since $f(x) = \sqrt{1 - x^2} \geqslant 0$, we can interpret this integral as the area under the curve $y = \sqrt{1 - x^2}$ from 0 to 1. But, since $y^2 = 1 - x^2$, we get $x^2 + y^2 = 1$, which shows that the graph of f is the quarter-circle with radius 1 in Figure 4. Therefore

$$\int_0^1 \sqrt{1 - x^2}\,dx = \tfrac{1}{4}\pi(1)^2 = \frac{\pi}{4}$$

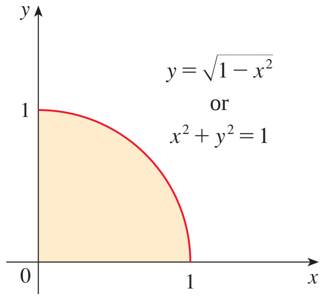


FIGURE 4

(In Section 8.3 we will be able to *prove* that the area of a circle of radius r is πr^2 by evaluating the integral $\int_0^r \sqrt{r^2 - x^2}\,dx$ using the techniques of Chapter 8.)

(b) The graph of $y = x - 1$ is the line with slope 1 shown in Figure 5. We compute the integral as the difference of the areas of the two triangles:

$$\int_0^3 (x - 1)\,dx = A_1 - A_2 = \tfrac{1}{2}(2 \cdot 2) - \tfrac{1}{2}(1 \cdot 1) = 1.5$$

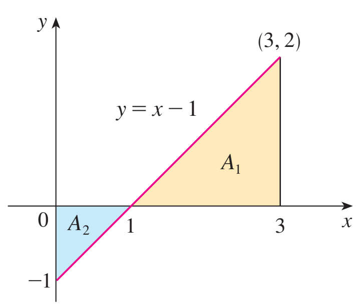


FIGURE 5

The integrals in Example 3 were simple to evaluate because we were able to express them in terms of areas of simple regions, but not all integrals are that easy. In fact, the integrals of some functions don't even exist. So the question arises: Which functions are integrable? A partial answer is given by the following theorem, which is proved in courses on advanced calculus.

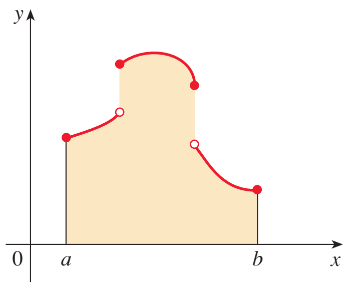


FIGURE 6
Discontinuous integrable function

> **4 Theorem** If f is either continuous or monotonic on $[a, b]$, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

If f is discontinuous at some points in $[a, b]$, then $\int_a^b f(x)\,dx$ might exist or it might not exist (see Exercises 70 and 71). If f has only a finite number of discontinuities and these are all jump discontinuities, then f is called **piecewise continuous** and it turns out that f is integrable. (See Figure 6.)

It can be shown that if f is integrable on $[a, b]$, then f must be a **bounded function** on $[a, b]$; that is, there exists a number M such that $|f(x)| \leq M$ for all x in $[a, b]$. Geometrically, this means that the graph of f lies between the horizontal lines $y = M$ and $y = -M$. In particular, if f has an infinite discontinuity at some point in $[a, b]$, then f is not bounded and is therefore not integrable. (See Exercise 70 and Figure 7.)

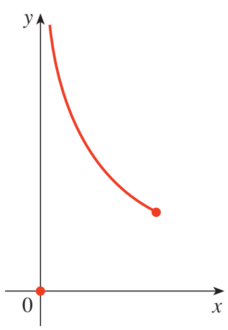


FIGURE 7
Nonintegrable function

If f is integrable on $[a, b]$, then the Riemann sums (3) must approach $\int_a^b f(x)\,dx$ as $\|P\| \to 0$ no matter how the partitions P are chosen and no matter how the points x_i^* are chosen in $[x_{i-1}, x_i]$. Therefore, if it is known beforehand that f is integrable on $[a, b]$ (for instance, if it is known that f is continuous or monotonic), then in calculating the value of an integral we are free to choose partitions P and points x_i^* in any way we like as long as $\|P\| \to 0$. For purposes of calculation, it is often convenient to take P to be a **regular partition**; that is, all the subintervals have the same length Δx. Then

$$\Delta x = \Delta x_1 = \Delta x_2 = \cdots = \Delta x_n = \frac{b - a}{n}$$

and $$x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \quad \ldots, \quad x_i = a + i\,\Delta x$$

If we choose x_i^* to be the right endpoint of the ith subinterval, then

$$x_i^* = x_i = a + i\,\Delta x = a + i\,\frac{b - a}{n}$$

Since $\|P\| = \Delta x = (b - a)/n$, we have $\|P\| \to 0$ as $n \to \infty$, so Definition 2 gives

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x \\ &= \lim_{n \to \infty} \sum_{i=1}^{n} f\left(a + i\,\frac{b - a}{n}\right)\frac{b - a}{n}\end{aligned}$$

Since $(b - a)/n$ does not depend on i, Theorem 1.2 allows us to take it in front of the sigma sign, and we have the following formula for calculating integrals.

> **5 Theorem** If f is integrable on $[a, b]$, then
>
> $$\int_a^b f(x)\,dx = \lim_{n \to \infty} \frac{b - a}{n} \sum_{i=1}^{n} f\left(a + i\,\frac{b - a}{n}\right)$$

EXAMPLE 4 Evaluate $\int_0^3 (x^3 - 5x)\,dx$.

SOLUTION Here we have $f(x) = x^3 - 5x$, $a = 0$, and $b = 3$. Since f is continuous, we know it is integrable and so Theorem 5 gives

$$\begin{aligned}\int_0^3 (x^3 - 5x)\,dx &= \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} f\left(\frac{3i}{n}\right) = \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - 5\left(\frac{3i}{n}\right)\right] \\ &= \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{45}{n^2}\sum_{i=1}^{n} i\right] \\ &= \lim_{n\to\infty}\left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{45}{n^2}\,\frac{n(n+1)}{2}\right\} \\ &= \lim_{n\to\infty}\left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - \frac{45}{2}\left(1+\frac{1}{n}\right)\right] \\ &= \tfrac{81}{4} - \tfrac{45}{2} = -\tfrac{9}{4} = -2.25\end{aligned}$$

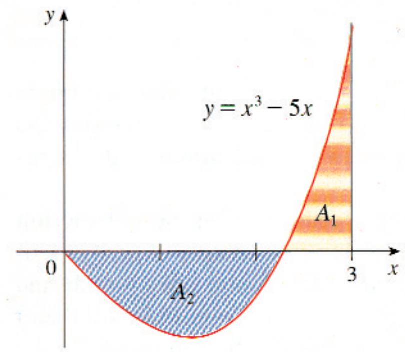


FIGURE 8

This integral cannot be interpreted as an area because f takes on both positive and negative values. But it can be interpreted as the difference of areas $A_1 - A_2$, where A_1 and A_2 are shown in Figure 8.

Figure 9 illustrates the calculation by showing the positive and negative terms in the right Riemann sum R_n for $n = 40$. The values in the table show the Riemann sums approaching the exact value of the integral, -2.25, as $n \to \infty$.

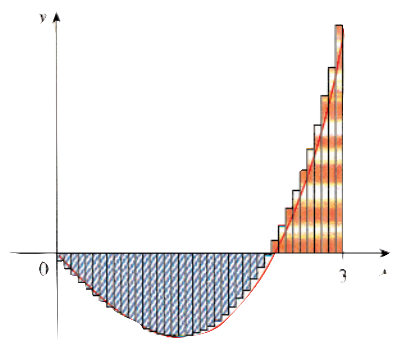


FIGURE 9

n	R_n
40	−1.7873
100	−2.0680
500	−2.2139
1000	−2.2320
5000	−2.2464

A much simpler method for evaluating the integral in Example 4 will be given in Section 5.4 after we have proved the Fundamental Theorem of Calculus.

The Midpoint Rule

We often choose the sample point x_i^* to be the right endpoint of the ith subinterval because it is convenient for computing the limit. But if the purpose is to find an *approximation* to an integral, it is usually better to choose x_i^* to be the midpoint of the interval, which we denote by $\bar{x}_i$. Any Riemann sum is an approximation to an integral, but if we use midpoints and a regular partition we get the following approximation:

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x = \Delta x\,[f(\bar{x}_1) + \cdots + f(\bar{x}_n)]$$

where $\Delta x = \dfrac{b-a}{n}$

and $\bar{x}_i = \tfrac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$

EXAMPLE 5 Use the Midpoint Rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x}\,dx$.

SOLUTION The partition points are 1, 1.2, 1.4, 1.6, 1.8, and 2.0, so the midpoints of the five intervals are 1.1, 1.3, 1.5, 1.7, and 1.9. The width of the intervals is $\Delta x = (2 - 1)/5 = \frac{1}{5}$, so the Midpoint Rule gives

$$\int_1^2 \frac{1}{x}\,dx \approx \Delta x\,[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)]$$

$$= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right)$$

$$\approx 0.691908$$

Since $f(x) = 1/x > 0$ for $1 \leqslant x \leqslant 2$, the integral represents an area and the approximation given by the Midpoint Rule is the sum of the areas of the rectangles shown in Figure 10.

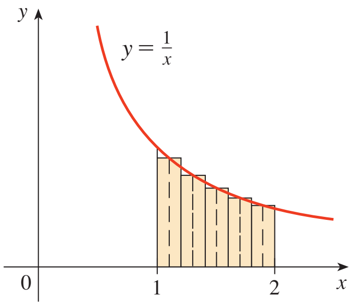


FIGURE 10

At the moment we don't know how accurate the approximation in Example 5 is, but in Section 8.7 we will learn a method for estimating the error involved in using the Midpoint Rule. At that time we will discuss other methods for approximating definite integrals.

If we apply the Midpoint Rule to the integral in Example 4, we get the picture in Figure 11. The approximation $M_{40} \approx -2.2563$ is much closer to the true value -2.25 than the right endpoint approximation, $R_{40} \approx -1.7873$ shown in Figure 9.

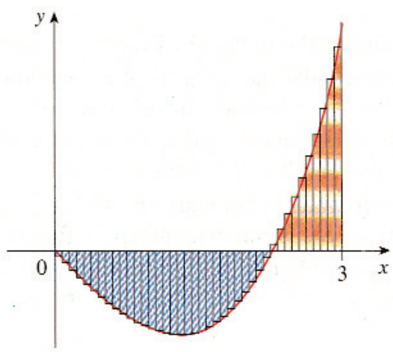


FIGURE 11
$M_{40} = -2.2563$

Properties of the Definite Integral

We now develop some basic properties of integrals that will help us to evaluate integrals in a simple manner.

Properties of the Integral Suppose that all of the following integrals exist. Then

1. $\int_a^b c\,dx = c(b - a)$, where c is any constant
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant
4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
5. $\int_a^b f(x)\,dx = \int_a^b f(x)\,dx + \int_a^b f(x)\,dx$

The proof of Property 1 is requested in Exercise 15. This property says that the integral of a constant function $f(x) = c$ is the constant times the length of the interval. If $c > 0$ and $a < b$, this is to be expected because $c(b - a)$ is the area of the shaded rectangle in Figure 12.

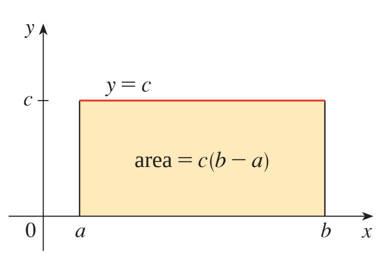


FIGURE 12
$\int_a^b c\,dx = c(b - a)$

Proof of Property 2 Since $\int_a^b [f(x) + g(x)]\,dx$ exists, we can compute it using a regular partition and choosing x_i^* to be the right endpoint of the ith subinterval, that is, $x_i^* = x_i$. Using the fact that the limit of a sum is the sum of the limits, we have

$$\begin{aligned}\int_a^b [f(x) + g(x)]\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) + g(x_i)]\,\Delta x \\ &= \lim_{n\to\infty} \left[\sum_{i=1}^{n} f(x_i)\,\Delta x + \sum_{i=1}^{n} g(x_i)\,\Delta x\right] \qquad \text{(by Theorem 1.2)} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x + \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\,\Delta x \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx\end{aligned}$$

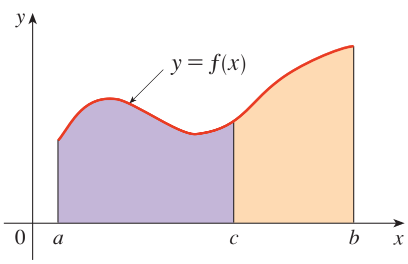


FIGURE 13

Property 2 says that the integral of a sum is the sum of the integrals. Property 3 can be proved in a similar manner (see Exercise 65) and says that the integral of a constant times a function is the constant times the integral of the function. In other words, a constant (but *only* a constant) can be taken in front of an integral sign. Property 4 is proved by writing $f - g = f + (-g)$ and using Properties 2 and 3 with $c = -1$.

Property 5 is somewhat more complicated and is proved at the end of this section, but for the case where $f(x) \geqslant 0$ and $a < c < b$, it can be seen from the geometric interpretation in Figure 13. For positive functions f, $\int_a^b f(x)\,dx$ is the total area under $y = f(x)$ from a to b, which is the sum of $\int_a^c f(x)\,dx$ (the area from a to c) and $\int_c^b f(x)\,dx$ (the area from c to b).

EXAMPLE 6 Use the properties of integrals and the results

$$\int_a^b x\,dx = \frac{b^2 - a^2}{2} \qquad \int_0^{\pi/2} \cos x\,dx = 1$$

(from Exercise 21 in this section and Example 4 in Section 2) to evaluate the following integrals.

(a) $\displaystyle\int_0^{\pi/2} (x + 3\cos x)\,dx$ (b) $\displaystyle\int_{-4}^{5} |x|\,dx$

SOLUTION

(a) Usiing Properties 2 and 3 of integrals, we get

$$\int_0^{\pi/2} (x + 3\cos x)\,dx = \int_0^{\pi/2} x\,dx + 3\int_0^{\pi/2} \cos x\,dx = \frac{\pi^2}{8} + 3$$

(b) Since

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

we use Property 5 to split the integral at 0:

$$\begin{aligned}\int_{-4}^{5} |x|\,dx &= \int_{-4}^{0} |x|\,dx + \int_0^5 |x|\,dx \\ &= \int_{-4}^{0} (-x)\,dx + \int_0^5 x\,dx = -\int_{-4}^{0} x\,dx + \int_0^5 x\,dx \\ &= -\tfrac{1}{2}[0^2 - (-4)^2] + \tfrac{1}{2}[5^2 - 0^2] = 20.5\end{aligned}$$

Notice that Properties 1–5 are true whether $a < b$, $a = b$, or $a > b$. The following properties, however, are true only if $a \leqslant b$.

Order Properties of the Integral Suppose the following integrals exist and $a \leqslant b$.

6. If $f(x) \geqslant 0$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant 0$.

7. If $f(x) \geqslant g(x)$ for $a \leqslant x \leqslant b$, then $\int_a^b f(x)\,dx \geqslant \int_a^b g(x)\,dx$.

8. If $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$, then

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$

9. $\left| \int_a^b f(x)\,dx \right| \leqslant \int_a^b |f(x)|\,dx$

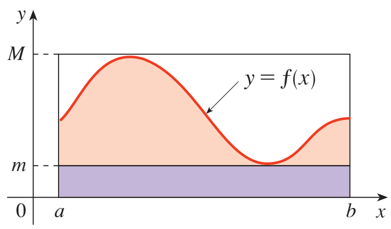


FIGURE 14

If $f(x) \geqslant 0$, then $\int_a^b f(x)\,dx$ represents the area under the graph of f, so the geometric interpretation of Property 6 is simply that areas are positive. But the property can be proved from the definition of an integral (Exercise 66). Property 7 says that a bigger function has a bigger integral. It follows from Properties 6 and 4 because $f - g \geqslant 0$.

Property 8 is illustrated by Figure 14 for the case where $f(x) \geqslant 0$. If f is continuous we could take m and M to be the absolute minimum and maximum values of f on the interval $[a, b]$. In this case Property 8 says that the area under the graph of f is greater than the area of the rectangle with height m and less than the area of the rectangle with height M.

Proof of Property 8 Since $m \leqslant f(x) \leqslant M$, Property 7 gives

$$\int_a^b m\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b M\,dx$$

Using Property 1 to evaluate the integrals on the left- and right-hand sides, we obtain

$$m(b - a) \leqslant \int_a^b f(x)\,dx \leqslant M(b - a)$$

The proof of Property 9 is left as Exercise 67.

EXAMPLE 7 Use Property 8 to estimate the value of $\int_1^4 \sqrt{x}\,dx$.

SOLUTION Since $f(x) = \sqrt{x}$ is an increasing function, its absolute minimum on $[1, 4]$ is $m = f(1) = 1$ and its absolute maximum on $[1, 4]$ is $M = f(4) = \sqrt{4} = 2$. Thus Property 8 gives

$$1(4 - 1) \leqslant \int_1^4 \sqrt{x}\,dx \leqslant 2(4 - 1)$$

or

$$3 \leqslant \int_1^4 \sqrt{x}\,dx \leqslant 6$$

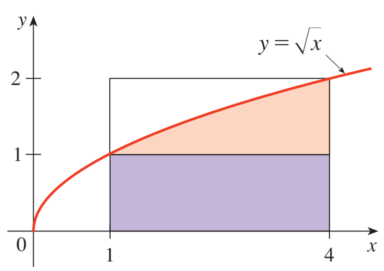


FIGURE 15

The result of Example 7 is illustrated in Figure 15. The area under $y = \sqrt{x}$ from 1 to 4 is greater than the area of the lower rectangle and less than the area of the large rectangle.

EXAMPLE 8 Show that $\int_1^4 \sqrt{1 + x^2}\,dx \geqslant 7.5$.

SOLUTION The minimum value of $f(x) = \sqrt{1 + x^2}$ on $[1, 4]$ is $m = f(1) = \sqrt{2}$, since f is increasing. Thus Property 8 gives

$$\int_1^4 \sqrt{1 + x^2}\,dx \geqslant \sqrt{2}(4 - 1) = 3\sqrt{2} \approx 4.24$$

This result is not good enough, so instead we use Property 7. Notice that

$$1 + x^2 > x^2 \quad \Rightarrow \quad \sqrt{1 + x^2} > \sqrt{x^2} = |x|$$

Since $|x| = x$ for $x > 0$, we have $\sqrt{1 + x^2} > x$ for $1 \leqslant x \leqslant 4$. Thus, by Property 7,

$$\int_1^4 \sqrt{1 + x^2}\, dx \geqslant \int_1^4 x\, dx = \tfrac{1}{2}(4^2 - 1^2) = 7.5$$

[Here we have used the fact that $\int_a^b x\, dx = (b^2 - a^2)/2$ from Exercise 21.]

Proof of Property 5 We first assume that $a < c < b$. Since we are assuming that $\int_a^b f(x)\, dx$ exists, we can compute it as a limit of Riemann sums using only partitions P that include c as one of the partition points. If P is such a partition, let P_1 be the corresponding partition of $[a, c]$ determined by those partition points of P that lie in $[a, c]$. Similarly, P_2 will denote the corresponding partition of $[c, b]$. Note that $\|P_1\| \leqslant \|P\|$ and $\|P_2\| \leqslant \|P\|$. Thus, if $\|P\| \to 0$, it follows that $\|P_1\| \to 0$ and $\|P_2\| \to 0$. If $\{x_i \mid 1 \leqslant i \leqslant n\}$ is the set of partition points for P and $n = k + m$, where k is the number of subintervals in $[a, c]$ and m is the number of subintervals in $[c, b]$, then $\{x_i \mid 1 \leqslant i \leqslant k\}$ is the set of partition points for P_1. If we write $t_j = x_{k+j}$ for the partition points to the right of c, then $\{t_i \mid 1 \leqslant j \leqslant m\}$ is the set of partition points for P_2. Thus we have

$$a = x_0 < x_1 < \cdots < x_k < x_{k+1} < \cdots < x_n = b$$

$$c < t_1 < \cdots < t_m = b$$

Choosing $x_i^* = x_i$ and letting $\Delta t_j = t_j - t_{j-1}$, we compute $\int_a^b f(x)\, dx$ as follows:

$$\begin{aligned}
\int_a^b f(x) &= \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i)\,\Delta x_i \\
&= \lim_{\|P\| \to 0} \left[\sum_{i=1}^{k} f(x_i)\,\Delta x_i + \sum_{i=k+1}^{n} f(x_i)\,\Delta x_i \right] \\
&= \lim_{\|P\| \to 0} \left[\sum_{i=1}^{k} f(x_i)\,\Delta x_i + \sum_{j=1}^{m} f(t_j)\,\Delta t_j \right] \\
&= \lim_{\|P_1\| \to 0} \sum_{i=1}^{k} f(x_i)\,\Delta x_i + \lim_{\|P_2\| \to 0} \sum_{j=1}^{m} f(t_j)\,\Delta t_j \\
&= \int_a^c f(x)\, dx + \int_c^b f(t)\, dt
\end{aligned}$$

Now suppose that $c < a < b$. By what we have already proved, we have

$$\int_a^b f(x) = \int_c^a f(x)\, dx + \int_a^b f(x)\, dx$$

Therefore

$$\begin{aligned}
\int_a^b f(x) &= -\int_c^a f(x)\, dx + \int_a^b f(x)\, dx \\
&= \int_a^c f(x)\, dx + \int_c^b f(x)\, dx
\end{aligned}$$

(See Note 6.) The proofs are similar for the remaining four orderings of a, b, and c.

3 Exercises

1–6 |||| You are given a function f, an interval, partition points that define a partition P, and points x_i^* in the ith subinterval. (a) Find $\|P\|$. (b) Find the Riemann sum (3).

1. $f(x) = 7 - 2x$, $[1, 5]$, $\{1, 1.6, 2.2, 3.0, 4.2, 5\}$, $x_i^* =$ midpoint

2. $f(x) = 3x - 1$, $[-2, 2]$, $\{-2, -1.2, -0.6, 0, 0.8, 1.6, 2\}$, $x_i^* =$ midpoint

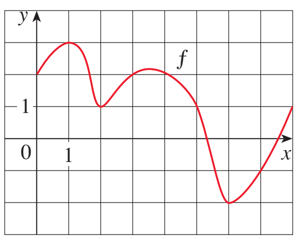
3. $f(x) = 2 - x^2$, $[-2, 2]$, $\{-2, -1.4, -1, 0, 0.8, 1.4, 2\}$, $x_i^* =$ right endpoint

4. $f(x) = x + x^2$, $[-2, 0]$, $\{-2, -1.5, -1, -0.7, -0.4, 0\}$, $x_i^* =$ left endpoint

5. $f(x) = x^3$, $[-1, 1]$, $\{-1, -0.5, 0, 0.5, 1\}$, $x_1^* = -1$, $x_2^* = -0.4$, $x_3^* = 0.2$, $x_4^* = 1$

6. $f(x) = \sin x$, $[-\pi/2, \pi]$, $\{-\pi/2, -1, 0, 1, 2, \pi\}$, $x_1^* = -1.5$, $x_2^* = -0.5$, $x_3^* = 0.5$, $x_4^* = 1.5$, $x_5^* = 3$

7. The graph of a function f is given. Estimate $\int_0^8 f(x)\,dx$ using four equal subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints.



8. The table gives the values of a function obtained from an experiment. Use them to estimate $\int_0^6 f(x)\,dx$ using three equal subintervals with (a) right endpoints, (b) left endpoints, and (c) midpoints. If the function is known to be a decreasing function, can you say whether your estimates are less than or greater than the exact value of the integral?

x	0	1	2	3	4	5	6
$f(x)$	9.3	9.0	8.3	6.5	2.3	−7.6	−10.5

9–12 |||| Use the Midpoint Rule with the given value of n to approximate the integral. Round the answer to four decimal places.

9. $\int_0^5 x^3\,dx, \quad n = 5$

10. $\int_1^3 \frac{1}{2x - 7}\,dx, \quad n = 4$

11. $\int_1^2 \sqrt{1 + x^2}\,dx, \quad n = 10$

12. $\int_0^{\pi/4} \tan x\,dx, \quad n = 4$

CAS **13.** If you have a CAS that evaluates midpoint approximations and graphs the corresponding rectangles (use `middlesum` and `middlebox` commands in Maple), check the answer to Exercise 11 and illustrate with a graph. Then repeat with $n = 20$ and $n = 30$.

14. With a programmable calculator or computer (see the instructions for Exercise 19 in Section 2), compute the left and right Riemann sums for the function $f(x) = \sqrt{1 + x^2}$ on the interval $[1, 2]$ with $n = 100$. Explain why these estimates show that

$$1.805 < \int_1^2 \sqrt{1 + x^2}\,dx < 1.815$$

Deduce that the approximation using the Midpoint Rule with $n = 10$ in Exercise 11 is accurate to two decimal places.

15–20 |||| Use Theorem 5 to evaluate the integral.

15. $\int_a^b c\,dx$

16. $\int_{-2}^7 (6 - 2x)\,dx$

17. $\int_1^4 (x^2 - 2)\,dx$

18. $\int_1^5 (2 + 3x - x^2)\,dx$

19. $\int_0^b (x^3 + 4x)\,dx$

20. $\int_0^1 (x^3 - 5x^4)\,dx$

21. Prove that $\int_a^b x\,dx = \dfrac{b^2 - a^2}{2}$.

22. Prove that $\int_a^b x^2\,dx = \dfrac{b^3 - a^3}{3}$.

23–28 |||| Evaluate the integral by interpreting it in terms of areas.

23. $\int_1^3 (1 + 2x)\,dx$

24. $\int_{-2}^2 \sqrt{4 - x^2}\,dx$

25. $\int_{-3}^0 \left(1 + \sqrt{9 - x^2}\right)dx$

26. $\int_{-1}^3 (2 - x)\,dx$

27. $\int_{-2}^2 \left(1 - |x|\right)dx$

28. $\int_0^3 |3x - 5|\,dx$

29–32 |||| Express the limit as a definite integral on the given interval.

29. $\lim_{\|P\| \to 0} \sum_{i=1}^n [2(x_i^*)^2 - 5x_i^*]\,\Delta x_i, \quad [0, 1]$

30. $\lim_{\|P\| \to 0} \sum_{i=1}^n \sqrt{x_i^*}\,\Delta x_i, \quad [1, 4]$

31. $\lim_{\|P\| \to 0} \sum_{i=1}^n \cos x_i\,\Delta x_i, \quad [0, \pi]$

32. $\lim_{\|P\| \to 0} \sum_{i=1}^n \frac{\tan x_i}{x_i}\,\Delta x_i, \quad [2, 4]$

33–35 |||| Express the limit as a definite integral.

33. $\lim_{n \to \infty} \sum_{i=1}^n \frac{i^4}{n^5}$ [*Hint:* Consider $f(x) = x^4$.]

34. $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{1 + (i/n)^2}$

35. $\lim_{n \to \infty} \sum_{i=1}^n \left[3\left(1 + \frac{2i}{n}\right)^5 - 6\right]\frac{2}{n}$

36. Evaluate $\int_1^1 x^2 \cos x\, dx$.

37. Given that $\int_4^9 \sqrt{x}\, dx = \frac{38}{3}$, what is $\int_9^4 \sqrt{t}\, dt$?

38. (a) Find an approximation to the integral $\int_0^4 (x^2 - 3x)\, dx$ using a Riemann sum with right endpoints and $n = 8$.
(b) Draw a diagram like Figure 1 to illustrate the approximation in part (a).
(c) Evaluate $\int_0^4 (x^2 - 3x)\, dx$.
(d) Interpret the integral in part (c) as a difference of areas and illustrate with a diagram like Figure 3.

39–44 Use the properties of integrals to evaluate each integral. You may assume from Section 2 that

$$\int_0^b \cos x\, dx = \sin b$$

and you may use the results of Exercises 21 and 22.

39. $\int_{-4}^{-1} \sqrt{3}\, dx$

40. $\int_3^6 (4 - 7x)\, dx$

41. $\int_1^4 (2x^2 - 3x + 1)\, dx$

42. $\int_0^1 (5 \cos x + 4x)\, dx$

43. $\int_{-1}^1 f(x)\, dx \quad \text{where } f(x) = \begin{cases} -2x & \text{if } -1 \leqslant x < 0 \\ 3x^2 & \text{if } 0 \leqslant x \leqslant 1 \end{cases}$

44. $\int_3^4 f(x)\, dx + \int_1^3 f(x)\, dx + \int_4^1 f(x)\, dx$

45–48 Write the given sum or difference as a single integral in the form $\int_a^b f(x)\, dx$.

45. $\int_1^3 f(x)\, dx + \int_3^6 f(x)\, dx + \int_6^{12} f(x)\, dx$

46. $\int_5^8 f(x)\, dx + \int_0^5 f(x)\, dx$

47. $\int_2^{10} f(x)\, dx - \int_2^7 f(x)\, dx$

48. $\int_{-3}^5 f(x)\, dx - \int_{-3}^0 f(x)\, dx + \int_5^6 f(x)\, dx$

49–54 Use the properties of integrals to verify the inequality without evaluating the integrals.

49. $\int_0^{\pi/4} \sin^3 x\, dx \leqslant \int_0^{\pi/4} \sin^2 x\, dx$

50. $\int_1^2 \sqrt{5 - x}\, dx \geqslant \int_1^2 \sqrt{x + 1}\, dx$

51. $\int_4^6 \frac{1}{x}\, dx \leqslant \int_4^6 \frac{1}{8 - x}\, dx$

52. $\frac{\pi}{6} \leqslant \int_{\pi/6}^{\pi/2} \sin x\, dx \leqslant \frac{\pi}{3}$

53. $2 \leqslant \int_{-1}^1 \sqrt{1 + x^2}\, dx \leqslant 2\sqrt{2}$

54. $\left| \int_0^{2\pi} f(x) \sin 2x\, dx \right| \leqslant \int_0^{2\pi} | f(x) |\, dx$

55–60 Use Property 8 to estimate the value of the integral.

55. $\int_1^2 \frac{1}{x}\, dx$

56. $\int_0^2 \sqrt{x^3 + 1}\, dx$

57. $\int_{-3}^0 (x^2 + 2x)\, dx$

58. $\int_{\pi/4}^{\pi/3} \cos x\, dx$

59. $\int_{-1}^1 \sqrt{1 + x^4}\, dx$

60. $\int_{\pi/4}^{3\pi/4} \sin^2 x\, dx$

61–64 Use properties of integrals, together with Exercises 21 and 22, to prove the inequality.

61. $\int_1^3 \sqrt{x^4 + 1}\, dx \geqslant 26/3$

62. $\int_2^5 \sqrt{x^2 - 1}\, dx \leqslant 10.5$

63. $\int_0^{\pi/2} x \sin x\, dx \leqslant \pi^2/8$

64. $\left| \int_0^{\pi} x^2 \cos x\, dx \right| \leqslant \pi^3/3$

65. Prove Property 3 of integrals.

66. Prove Property 6 of integrals.

67. Prove Property 9 of integrals.
[*Hint:* $-| f(x) | \leqslant f(x) \leqslant | f(x) |$.]

68. Suppose that f is continuous on $[a, b]$ and $f(x) > 0$ for all x in $[a, b]$. Prove that $\int_a^b f(x)\, dx > 0$. [*Hint:* Use the Extreme Value Theorem and Property 8.]

69. Which of the following functions are integrable on the interval $[0, 2]$?
(a) $f(x) = x^2 \sin x$
(b) $f(x) = \sec x$
(c) $f(x) = \begin{cases} x + 1 & \text{if } 0 \leqslant x < 1 \\ 2 - x & \text{if } 1 \leqslant x \leqslant 2 \end{cases}$
(d) $f(x) = \begin{cases} (x + 1)^{-2} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$

70. Let

$$f(t) = \begin{cases} \dfrac{1}{x} & \text{if } 0 < x \leqslant 1 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Show that f is not continuous on $[0, 1]$.
(b) Show that f is unbounded on $[0, 1]$.
(c) Show that $\int_0^1 f(x)\, dx$ does not exist, that is, f is not integrable on $[0, 1]$. [*Hint:* Show that the first term in the Riemann sum, $f(x_1^*)\, \Delta x_1$, can be made arbitrarily large.]

71. Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is bounded but not integrable on $[a, b]$. [*Hint:* Show that, no matter how small $\|P\|$ is, some Riemann sums are 0 whereas others are equal to $b - a$.]

72. Evaluate $\int_1^2 x^3\,dx$ using a partition of $[1, 2]$ by points of a geometric progression: $x_0 = 1$, $x_1 = 2^{1/n}$, $x_2 = 2^{2/n}, \ldots,$ $x_i = 2^{i/n}, \ldots, x_n = 2^{n/n} = 2$. Take $x_i^* = x_i$ and use the formula in Exercise 47 in Section 1 for the sum of a geometric series.

73. Find $\int_1^2 x^{-2}\,dx$. *Hint:* Use a regular partition but choose x_i^* to be the geometric mean of x_{i-1} and x_i $\left(x_i^* = \sqrt{x_{i-1}x_i}\right)$ and use the identity

$$\frac{1}{m(m+1)} = \frac{1}{m} - \frac{1}{m+1}$$

74. (a) Draw the graph of the function $f(x) = \cos(x^2)$ in the viewing rectangle $[0, 2]$ by $[-1, 1]$.

(b) If we define a new function g by $g(x) = \int_0^x \cos(t^2)\,dt$, then $g(x)$ is the area under the graph of f from 0 to x [until $f(x)$ becomes negative, at which point $g(x)$ becomes a difference of areas.] Use the graph of f from part (a) to estimate the value of $g(x)$ when $x = 0, 0.2, 0.4, 0.6, \ldots$ up to $x = 2$. At what value of x does $g(x)$ start to decrease?

(c) Use the information from part (a) to sketch a rough graph of g.

(d) Sketch a more accurate graph of g by using your calculator or computer to estimate $g(0.2), g(0.4), \ldots$. (Use the `integration` command, if available, or the Midpoint Rule.)

(e) Use your graph of g from part (d) to sketch the graph of g' using the interpretation of $g'(x)$ as the slope of a tangent line. How does the graph of g' compare with the graph of f?