CHALLENGE PROBLEMS

CHAPTER 1

A Click here for answers. S Click here for solutions.

1. Evaluate $\lim_{x \to 1} \dfrac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$.

2. Find numbers a and b such that $\lim_{x \to 0} \dfrac{\sqrt{ax + b} - 2}{x} = 1$.

3. Evaluate $\lim_{x \to 0} \dfrac{|2x - 1| - |2x + 1|}{x}$.

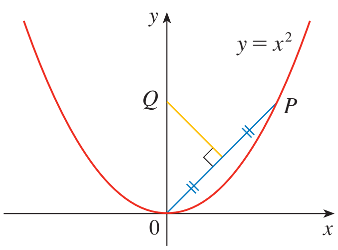

FIGURE FOR PROBLEM 4

4. The figure shows a point P on the parabola $y = x^2$ and the point Q where the perpendicular bisector of OP intersects the y-axis. As P approaches the origin along the parabola, what happens to Q? Does it have a limiting position? If so, find it.

5. If $[\![x]\!]$ denotes the greatest integer function, find $\lim_{x \to \infty} x/[\![x]\!]$.

6. Sketch the region in the plane defined by each of the following equations.
(a) $[\![x]\!]^2 + [\![y]\!]^2 = 1$ (b) $[\![x]\!]^2 - [\![y]\!]^2 = 3$ (c) $[\![x + y]\!]^2 = 1$ (d) $[\![x]\!] + [\![y]\!] = 1$

7. Find all values of a such that f is continuous on $\mathbb{R}$:

$$f(x) = \begin{cases} x + 1 & \text{if } x \le a \\ x^2 & \text{if } x > a \end{cases}$$

8. A **fixed point** of a function f is a number c in its domain such that $f(c) = c$. (The function doesn't move c; it stays fixed.)
(a) Sketch the graph of a continuous function with domain $[0, 1]$ whose range also lies in $[0, 1]$. Locate a fixed point of f.
(b) Try to draw the graph of a continuous function with domain $[0, 1]$ and range in $[0, 1]$ that does *not* have a fixed point. What is the obstacle?
(c) Use the Intermediate Value Theorem to prove that any continuous function with domain $[0, 1]$ and range a subset of $[0, 1]$ must have a fixed point.

9. If $\lim_{x \to a} [f(x) + g(x)] = 2$ and $\lim_{x \to a} [f(x) - g(x)] = 1$, find $\lim_{x \to a} f(x)g(x)$.

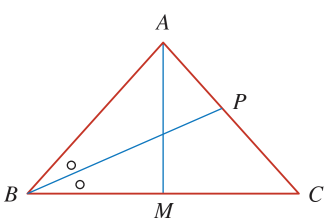

FIGURE FOR PROBLEM 10

10. (a) The figure shows an isosceles triangle ABC with $\angle B = \angle C$. The bisector of angle B intersects the side AC at the point P. Suppose that the base BC remains fixed but the altitude $|AM|$ of the triangle approaches 0, so A approaches the midpoint M of BC. What happens to P during this process? Does it have a limiting position? If so, find it.
(b) Try to sketch the path traced out by P during this process. Then find the equation of this curve and use this equation to sketch the curve.

11. (a) If we start from 0° latitude and proceed in a westerly direction, we can let $T(x)$ denote the temperature at the point x at any given time. Assuming that T is a continuous function of x, show that at any fixed time there are at least two diametrically opposite points on the equator that have exactly the same temperature.
(b) Does the result in part (a) hold for points lying on any circle on Earth's surface?
(c) Does the result in part (a) hold for barometric pressure and for altitude above sea level?

ANSWERS

S Solutions **1.** $\frac{2}{3}$ **3.** -4 **5.** 1 **7.** $a = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$ **9.** $\frac{3}{4}$ **11.** (b) Yes (c) Yes; no

SOLUTIONS

E Exercises

1. Let $t = \sqrt[6]{x}$, so $x = t^6$. Then $t \to 1$ as $x \to 1$, so

$$\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1} = \lim_{t\to 1} \frac{t^2-1}{t^3-1} = \lim_{t\to 1} \frac{(t-1)(t+1)}{(t-1)(t^2+t+1)} = \lim_{t\to 1} \frac{t+1}{t^2+t+1} = \frac{1+1}{1^2+1+1} = \frac{2}{3}$$

Another method: Multiply both the numerator and the denominator by $(\sqrt{x}+1)\left(\sqrt[3]{x^2}+\sqrt[3]{x}+1\right)$.

3. For $-\frac{1}{2} < x < \frac{1}{2}$, we have $2x-1<0$ and $2x+1>0$, so $|2x-1| = -(2x-1)$ and $|2x+1| = 2x+1$.

Therefore, $\displaystyle\lim_{x\to 0} \frac{|2x-1|-|2x+1|}{x} = \lim_{x\to 0} \frac{-(2x-1)-(2x+1)}{x} = \lim_{x\to 0} \frac{-4x}{x} = \lim_{x\to 0}(-4) = -4.$

5. Since $[\![x]\!] \le x < [\![x]\!]+1$, we have $\dfrac{[\![x]\!]}{[\![x]\!]} \le \dfrac{x}{[\![x]\!]} < \dfrac{[\![x]\!]+1}{[\![x]\!]} \quad\Rightarrow\quad 1 \le \dfrac{x}{[\![x]\!]} < 1 + \dfrac{1}{[\![x]\!]}$ for $x \ge 1$. As $x \to \infty$,

$[\![x]\!] \to \infty$, so $\dfrac{1}{[\![x]\!]} \to 0$ and $1 + \dfrac{1}{[\![x]\!]} \to 1$. Thus, $\displaystyle\lim_{x\to\infty} \frac{x}{[\![x]\!]} = 1$ by the Squeeze Theorem.

7. f is continuous on $(-\infty, a)$ and (a, ∞). To make f continuous on $\mathbb{R}$, we must have continuity at a.

Thus, $\displaystyle\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) \quad\Rightarrow\quad \lim_{x\to a^+} x^2 = \lim_{x\to a^-}(x+1) \quad\Rightarrow\quad a^2 = a+1 \quad\Rightarrow\quad a^2-a-1=0 \quad\Rightarrow$

[by the quadratic formula] $\quad a = \left(1 \pm \sqrt{5}\right)/2 \approx 1.618$ or -0.618.

9. $\displaystyle\lim_{x\to a} f(x) = \lim_{x\to a}\left(\tfrac{1}{2}[f(x)+g(x)] + \tfrac{1}{2}[f(x)-g(x)]\right) = \tfrac{1}{2}\lim_{x\to a}[f(x)+g(x)] + \tfrac{1}{2}\lim_{x\to a}[f(x)-g(x)]$

$\qquad = \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 1 = \frac{3}{2}$, and

$\displaystyle\lim_{x\to a} g(x) = \lim_{x\to a}\Big([f(x)+g(x)] - f(x)\Big) = \lim_{x\to a}[f(x)+g(x)] - \lim_{x\to a} f(x) = 2 - \tfrac{3}{2} = \tfrac{1}{2}.$

So $\displaystyle\lim_{x\to a}[f(x)g(x)] = \Big[\lim_{x\to a} f(x)\Big]\Big[\lim_{x\to a} g(x)\Big] = \tfrac{3}{2}\cdot\tfrac{1}{2} = \tfrac{3}{4}.$

Another solution: Since $\displaystyle\lim_{x\to a}[f(x)+g(x)]$ and $\displaystyle\lim_{x\to a}[f(x)-g(x)]$ exist, we must have

$\displaystyle\lim_{x\to a}[f(x)+g(x)]^2 = \Big(\lim_{x\to a}[f(x)+g(x)]\Big)^2$ and $\displaystyle\lim_{x\to a}[f(x)-g(x)]^2 = \Big(\lim_{x\to a}[f(x)-g(x)]\Big)^2$, so

$\displaystyle\lim_{x\to a}[f(x)\,g(x)] = \lim_{x\to a}\tfrac{1}{4}\left([f(x)+g(x)]^2 - [f(x)-g(x)]^2\right)$ [because all of the f^2 and g^2 cancel]

$$= \tfrac{1}{4}\Big(\lim_{x\to a}[f(x)+g(x)]^2 - \lim_{x\to a}[f(x)-g(x)]^2\Big) = \tfrac{1}{4}\left(2^2-1^2\right) = \tfrac{3}{4}.$$

11. (a) Consider $G(x) = T(x+180^\circ) - T(x)$. Fix any number a. If $G(a) = 0$, we are done: Temperature at a = Temperature at $a+180^\circ$. If $G(a) > 0$, then

$$G(a+180^\circ) = T(a+360^\circ) - T(a+180^\circ) = T(a) - T(a+180^\circ) = -G(a) < 0$$

Also, G is continuous since temperature varies continuously. So, by the Intermediate Value Theorem, G has a zero on the interval $[a, a+180^\circ]$. If $G(a) < 0$, then a similar argument applies.

(b) Yes. The same argument applies.

(c) The same argument applies for quantities that vary continuously, such as barometric pressure. But one could argue that altitude above sea level is sometimes discontinuous, so the result might not always hold for that quantity.