

4.5 THE SUBSTITUTION RULE

EXAMPLE A Find $\int \sqrt{1+x^2} x^5 dx$.

SOLUTION An appropriate substitution becomes more obvious if we factor x^5 as $x^4 \cdot x$. Let $u = 1 + x^2$. Then $du = 2x dx$, so $x dx = du/2$. Also $x^2 = u - 1$, so $x^4 = (u - 1)^2$:

$$\begin{aligned} \int \sqrt{1+x^2} x^5 dx &= \int \sqrt{1+x^2} x^4 \cdot x dx \\ &= \int \sqrt{u} (u-1)^2 \frac{du}{2} = \frac{1}{2} \int \sqrt{u} (u^2 - 2u + 1) du \\ &= \frac{1}{2} \int (u^{5/2} - 2u^{3/2} + u^{1/2}) du \\ &= \frac{1}{2} \left(\frac{2}{7} u^{7/2} - 2 \cdot \frac{2}{3} u^{3/2} + \frac{2}{3} u^{3/2} \right) + C \\ &= \frac{1}{7} (1+x^2)^{7/2} - \frac{2}{5} (1+x^2)^{5/2} + \frac{1}{3} (1+x^2)^{3/2} + C \quad \blacksquare \end{aligned}$$

EXAMPLE B Evaluate $\int_0^4 \sqrt{2x+1} dx$ using (6).

SOLUTION Using the substitution from Solution 1 of Example 2, we have $u = 2x + 1$ and $dx = du/2$. To find the new limits of integration we note that

$$\text{when } x = 0, u = 2(0) + 1 = 1 \quad \text{and} \quad \text{when } x = 4, u = 2(4) + 1 = 9$$

Therefore

$$\begin{aligned} \int_0^4 \sqrt{2x+1} dx &= \int_1^9 \frac{1}{2} \sqrt{u} du \\ &= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_1^9 \\ &= \frac{1}{3} (9^{3/2} - 1^{3/2}) = \frac{26}{3} \end{aligned}$$

■ The geometric interpretation of Example B is shown in Figure 1. The substitution $u = 2x + 1$ stretches the interval $[0, 4]$ by a factor of 2 and translates it to the right by 1 unit. The Substitution Rule shows that the two areas are equal.

Observe that when using (6) we do not return to the variable x after integrating. We simply evaluate the expression in u between the appropriate values of u . ■

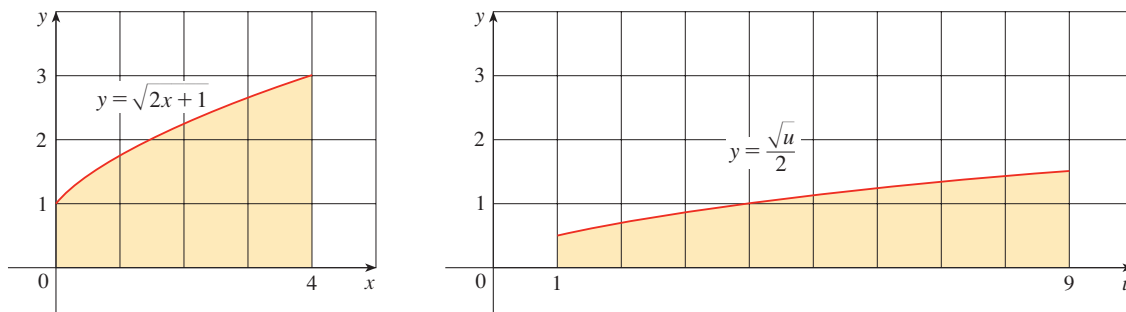


FIGURE 1