5.3 | THE NATURAL EXPONENTIAL FUNCTION

EXAMPLE A Use the first and second derivatives of $f(x) = e^{1/x}$, together with asymptotes, to sketch its graph.

SOLUTION Notice that the domain of f is $\{x \mid x \neq 0\}$, so we check for vertical asymptotes by computing the left and right limits as $x \to 0$. As $x \to 0^+$, we know that $t = 1/x \to \infty$, so

$$\lim_{x \to 0^+} e^{1/x} = \lim_{t \to \infty} e^t = \infty$$

and this shows that $x = 0$ is a vertical asymptote. As $x \to 0^-$, we have $t = 1/x \to -\infty$, so

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$

As $x \to \pm\infty$, we have $1/x \to 0$ and so

$$\lim_{x \to \pm\infty} e^{1/x} = e^0 = 1$$

This shows that $y = 1$ is a horizontal asymptote.

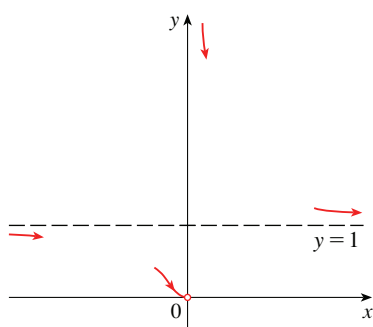Now let's compute the derivative. The Chain Rule gives

$$f'(x) = -\frac{e^{1/x}}{x^2}$$

Since $e^{1/x} > 0$ and $x^2 > 0$ for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$. Thus, f is decreasing on $(-\infty, 0)$ and on $(0, \infty)$. There is no critical number, so the function has no maximum or minimum. The second derivative is
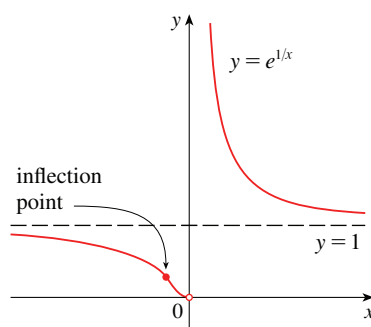
$$f''(x) = -\frac{x^2 e^{1/x}(-1/x^2) - e^{1/x}(2x)}{x^4} = \frac{e^{1/x}(2x + 1)}{x^4}$$

Since $e^{1/x} > 0$ and $x^4 > 0$, we have $f''(x) > 0$ when $x > -\frac{1}{2}$ ($x \neq 0$) and $f''(x) < 0$ when $x < -\frac{1}{2}$. So the curve is concave downward on $(-\infty, -\frac{1}{2})$ and concave upward on $(-\frac{1}{2}, 0)$ and on $(0, \infty)$. The inflection point is $(-\frac{1}{2}, e^{-2})$.
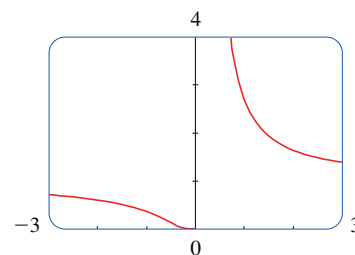
To sketch the graph of f we first draw the horizontal asymptote $y = 1$ (as a dashed line), together with the parts of the curve near the asymptotes in a preliminary sketch [Figure 1(a)]. These parts reflect the information concerning limits and the fact that f is decreasing on both $(-\infty, 0)$ and $(0, \infty)$. Notice that we have indicated that $f(x) \to 0$ as $x \to 0^-$ even though $f(0)$ does not exist. In Figure 1(b) we finish the sketch by incorporating the information concerning concavity and the inflection point. In Figure 1(c) we check our work with a graphing device.

TEC In Module 5.3 you can practice using graphical information about f' to determine the shape of the graph of f.

(a) Preliminary sketch

(b) Finished sketch

(c) Computer confirmation

FIGURE 1

EXAMPLE B A population of honeybees raised in an apiary started with 50 bees at time $t = 0$ and was modeled by the function

$$P(t) = \frac{75{,}200}{1 + 1503e^{-0.5932t}}$$

where t is the time in weeks, $0 \leq t \leq 25$. Use a graph to estimate the time at which the bee population was growing fastest. Then use derivatives to give a more accurate estimate.

SOLUTION The population grows fastest when the population curve $y = P(t)$ has the steepest tangent line. From the graph of P in Figure 2, we estimate that the steepest tangent occurs when $t \approx 12$, so the bee population was growing most rapidly after about 12 weeks.

For a better estimate we calculate the derivative $P'(t)$, which is the rate of increase of the bee population:

$$P'(t) = -\frac{67{,}046{,}785.92e^{-0.5932t}}{(1 + 1503e^{-0.5932t})^2}$$

We graph P' in Figure 3 and observe that P' has its maximum value when $t \approx 12.3$.

To get a still better estimate we note that f' has its maximum value when f' changes from increasing to decreasing. This happens when f changes from concave upward to concave downward, that is, when f has an inflection point. So we ask a CAS to compute the second derivative:

$$P''(t) \approx \frac{119555093144e^{-1.1864t}}{(1 + 1503e^{-0.5932t})^3} - \frac{39772153e^{-0.5932t}}{(1 + 1503e^{-0.5932t})^2}$$

We could plot this function to see where it changes from positive to negative, but instead let's have the CAS solve the equation $P''(t) = 0$. It gives the answer $t \approx 12.3318$. ∎
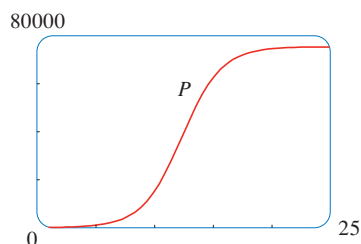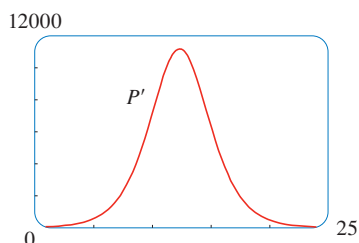

FIGURE 2

FIGURE 3