7.2 VOLUMES

A Click here for answers.

S Click here for solutions.

1–5 ▪ Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

1. $y^2 = x^3$, $x = 4$, $y = 0$; about the x-axis

2. $x + y = 1$, $x = 0$, $y = 0$; about the x-axis

3. $y = x^2$, $y = 4$, $x = 0$, $x = 2$; about the y-axis

4. $y = x^2 + 1$, $y = 3 - x^2$; about the x-axis

5. $y = 2x - x^2$, $y = 0$, $x = 0$, $x = 1$; about the y-axis

6–13 ▪ Find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis.

6. $y = x^2 - 1$, $y = 0$, $x = 0$, $x = 2$

7. $y = -1/x$, $y = 0$, $x = 1$, $x = 3$

8. $y = e^x$, $y = 0$, $x = 0$, $x = 1$

9. $y = 1/\sqrt{x + 1}$, $y = 0$, $x = 0$, $x = 1$

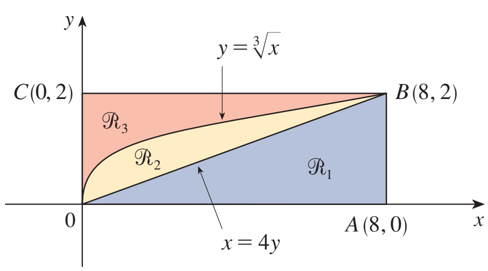
10. $y = \sec x$, $y = 1$, $x = -1$, $x = 1$

11. $y = \cos x$, $y = \sin x$, $x = 0$, $x = \pi/4$

12. $y = |x + 2|$, $y = 0$, $x = -3$, $x = 0$

13. $y = [\![x]\!]$, $x = 1$, $x = 6$, $y = 0$

14–25 ▪ Refer to the figure and find the volume generated by rotating the given region about the given line.



14. $\mathcal{R}_1$ about OA

15. $\mathcal{R}_1$ about OC

16. $\mathcal{R}_1$ about AB

17. $\mathcal{R}_1$ about BC

18. $\mathcal{R}_2$ about OA

19. $\mathcal{R}_2$ about OC

20. $\mathcal{R}_2$ about BC

21. $\mathcal{R}_2$ about AB

22. $\mathcal{R}_3$ about OA

23. $\mathcal{R}_3$ about OC

24. $\mathcal{R}_3$ about BC

25. $\mathcal{R}_3$ about AB

26–30 ▪ Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

26. $y = \ln x$, $y = 1$, $x = 1$; about the x-axis

27. $y = \sqrt{x - 1}$, $y = 0$, $x = 5$; about the y-axis

28. $x - y = 1$, $y = (x - 4)^2 + 1$; about $y = 7$

29. $y = \cos x$, $y = 0$, $x = 0$, $x = \pi/2$; about $y = 1$

30. $y = \cos x$, $y = 0$, $x = 0$, $x = \pi/2$; about $y = -1$

31–32 ▪ Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the volume of the solid obtained by rotating about the x-axis the region bounded by these curves.

31. $y = x^2$, $y = \sqrt{x + 1}$

32. $y = x^4$, $y = 3x - x^3$

33–34 ▪ Sketch and find the volume of the solid obtained by rotating the region under the graph of f about the x-axis.

33. $f(x) = \begin{cases} 3 & \text{if } 0 \leqslant x \leqslant 1 \\ 1 & \text{if } 1 < x < 4 \\ 3 & \text{if } 4 \leqslant x \leqslant 5 \end{cases}$

34. $f(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leqslant x < 1 \\ x^2 - 2x + 2 & \text{if } 1 \leqslant x \leqslant 2 \end{cases}$

35–40 ▪ Each integral represents the volume of a solid. Describe the solid.

35. $\pi \int_0^{\pi/4} \tan^2 x\, dx$

36. $\pi \int_1^2 y^6\, dy$

37. $\pi \int_0^1 (y - y^2)\, dy$

38. $\pi \int_0^4 [16 - (x - 2)^4]\, dx$

39. $\pi \int_0^1 [(5 - 2x^2)^2 - (5 - 2x)^2]\, dx$

40. $\pi \int_{\pi/4}^{\pi/2} [(2 + \sin x)^2 - (2 + \cos x)^2]\, dx$

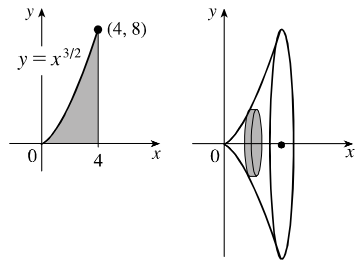
41. The base of S is the triangular region with vertices (0, 0), (2, 0), and (0, 1). Cross-sections perpendicular to the x-axis are semicircles. Find the volume of S.

7.2 ANSWERS

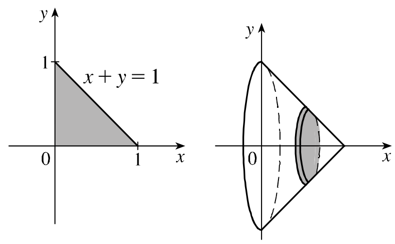
E Click here for exercises.

S Click here for solutions.

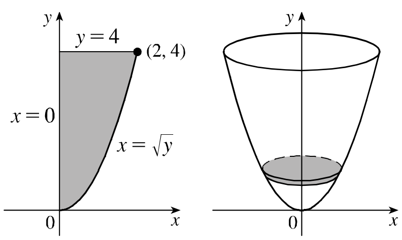
1. 64π



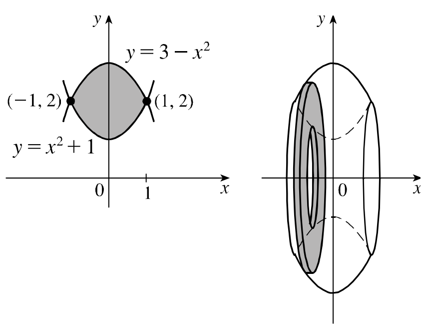
2. $\frac{\pi}{3}$



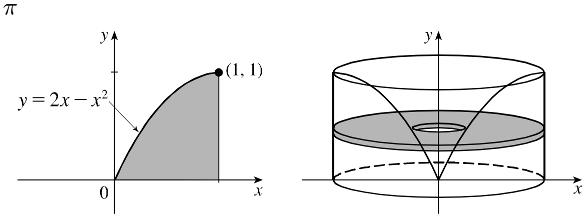
3. 8π



4. $\frac{32}{3}\pi$



5. $\frac{5}{6}\pi$



6. $\frac{46}{15}\pi$

7. $\frac{2}{3}\pi$

8. $\frac{\pi}{2}\left(e^2 - 1\right)$

9. $\pi \ln 2$

10. $2\pi\,(\tan 1 - 1)$

11. $\frac{\pi}{2}$

12. 3π

13. 55π

14. $\frac{32}{3}\pi$

15. $\frac{256}{3}\pi$

16. $\frac{128}{3}\pi$

17. $\frac{64}{3}\pi$

18. $\frac{128}{15}\pi$

19. $\frac{512}{21}\pi$

20. $\frac{112}{15}\pi$

21. $\frac{832}{21}\pi$

22. $\frac{64}{5}\pi$

23. $\frac{128}{7}\pi$

24. $\frac{16}{5}\pi$

25. $\frac{320}{7}\pi$

26. $V = \pi \int_1^e \left[1^2 - (\ln x)^2\right] dx$

27. $V = \pi \int_0^2 \left(24 - y^4 - 2y^2\right) dy$

28. $V = \pi \int_3^6 \left(x^4 - 16x^3 + 83x^2 - 144x + 36\right) dx$

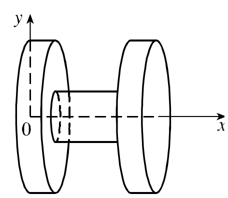
29. $V = \pi \int_0^{\pi/2} \left(2\cos x - \cos^2 x\right) dx$

30. $V = \pi \int_0^{\pi/2} \left(2\cos x + \cos^2 x\right) dx$

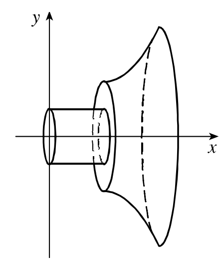
31. 5.80

32. 6.74

33. 21π



34. $\frac{127\pi}{60}$



35. Solid obtained by rotating the region under the curve $y = \tan x$, from $x = 0$ to $x = \frac{\pi}{4}$, about the x-axis

36. Solid obtained by rotating the region bounded by the curve $x = y^3$ and the lines $y = 1$, $y = 2$, and $x = 0$ about the y-axis

37. Solid obtained by rotating the region between the curves $x = y$ and $x = \sqrt{y}$ about the y-axis

38. Solid obtained by rotating the region bounded by the curve $y = (x - 2)^2$ and the line $y = 4$ about the x-axis

39. Solid obtained by rotating the region between the curves $y = 5 - 2x^2$ and $y = 5 - 2x$ about the x-axis. *Or:* Solid obtained by rotating the region bounded by the curves $y = 2x$ and $y = 2x^2$ about the line $y = 5$

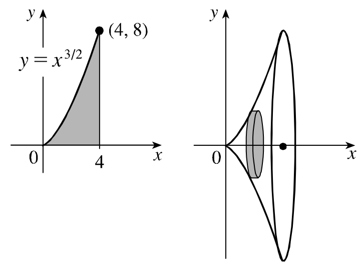
40. Solid obtained by rotating the region bounded by the curves $y = 2 + \cos x$ and $y = 2 + \sin x$ and the line $x = \frac{\pi}{2}$ about the x-axis

41. $\frac{\pi}{12}$

7.2 SOLUTIONS

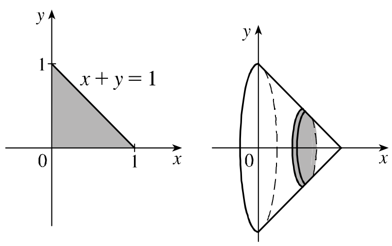
E Click here for exercises.

1. $V = \int_0^4 \pi \left(x^{3/2}\right)^2 \, dx = \pi \left[\frac{1}{4}x^4\right]_0^4 = 64\pi$

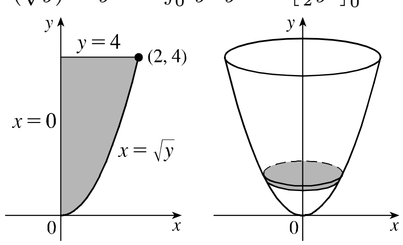


2. $V = \int_0^1 \pi \left(-x+1\right)^2 \, dx = \pi \int_0^1 \left(x^2 - 2x + 1\right) dx$

$= \pi \left[\frac{1}{3}x^3 - x^2 + x\right]_0^1 = \pi \left(\frac{1}{3} - 1 + 1\right) = \frac{\pi}{3}$



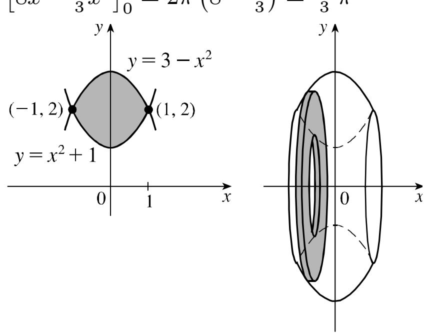
3. $V = \int_0^4 \pi \left(\sqrt{y}\right)^2 \, dy = \pi \int_0^4 y \, dy = \pi \left[\frac{1}{2}y^2\right]_0^4 = 8\pi$



4. $V = \pi \int_{-1}^1 \left[\left(3 - x^2\right)^2 - \left(x^2 + 1\right)^2\right] dx$

$= \pi \int_{-1}^1 \left(8 - 8x^2\right) dx = 2\pi \int_0^1 \left(8 - 8x^2\right) dx$

$= 2\pi \left[8x - \frac{8}{3}x^3\right]_0^1 = 2\pi \left(8 - \frac{8}{3}\right) = \frac{32}{3}\pi$

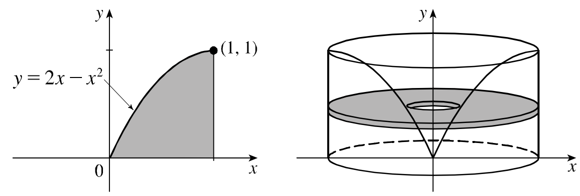


5. $V = \pi \int_0^1 \left[1^2 - \left(1 - \sqrt{1-y}\right)^2\right] dy$

$= \pi \int_0^1 \left(2\sqrt{1-y} - 1 + y\right) dy$

$= \pi \left[-\frac{4}{3}\left(1-y\right)^{3/2} - y + \frac{1}{2}y^2\right]_0^1$

$= \pi \left[\left(0 - 1 + \frac{1}{2}\right) - \left(-\frac{4}{3} - 0 + 0\right)\right] = \frac{5}{6}\pi$



6. $V = \pi \int_0^2 \left(x^2 - 1\right)^2 \, dx = \pi \int_0^2 \left(x^4 - 2x^2 + 1\right) dx$

$= \pi \left[\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right]_0^2 = \pi \left(\frac{32}{5} - \frac{16}{3} + 2\right) = \frac{46}{15}\pi$

7. $V = \pi \int_1^3 \left(1/x\right)^2 \, dx = \pi \left[-1/x\right]_1^3 = \pi \left(-\frac{1}{3} + 1\right) = \frac{2}{3}\pi$

8. $V = \int_0^1 \pi \left(e^x\right)^2 \, dx = \int_0^1 \pi e^{2x} \, dx = \frac{1}{2}\left[\pi e^{2x}\right]_0^1$

$= \frac{\pi}{2}\left(e^2 - 1\right)$

9. The cross-sectional area is $\pi \left(1/\sqrt{x+1}\right)^2 = \pi / \left(x+1\right)$. Therefore, the volume is $\displaystyle\int_0^1 \frac{\pi}{x+1} \, dx = \pi \left[\ln\left(x+1\right)\right]_0^1 = \pi \ln 2 - \ln 1 = \pi \ln 2.$

10. $V = \pi \int_{-1}^1 \left(\sec^2 x - 1^2\right) dx = \pi \left[\tan x - x\right]_{-1}^1$

$= \pi \left[\left(\tan 1 - 1\right) - \left(-\tan 1 + 1\right)\right] = 2\pi \left(\tan 1 - 1\right)$

11. $V = \pi \int_0^{\pi/4} \left(\cos^2 x - \sin^2 x\right) dx$

$= \frac{\pi}{2} \int_0^{\pi/4} \cos 2x \left(2\, dx\right) = \frac{\pi}{2} \left[\sin 2x\right]_0^{\pi/4}$

$= \frac{\pi}{2}\left(1 - 0\right) = \frac{\pi}{2}$

12. $V = \pi \int_{-3}^{-2} \left(-x - 2\right)^2 \, dx + \pi \int_{-2}^0 \left(x + 2\right)^2 \, dx$

$= \pi \int_{-3}^0 \left(x + 2\right)^2 \, dx = \left[\frac{\pi}{3}\left(x + 2\right)^3\right]_{-3}^0$

$= \frac{\pi}{3}\left[8 - \left(-1\right)\right] = 3\pi$

13. $V = \pi \int_1^2 1^2 \, dx + \pi \int_2^3 2^2 \, dx + \pi \int_3^4 3^2 \, dx$

$+ \pi \int_4^5 4^2 \, dx + \pi \int_5^6 5^2 \, dx$

$= \pi \cdot 1 + \pi \cdot 4 + \pi \cdot 9 + \pi \cdot 16 + \pi \cdot 25 = 55\pi$

14. $V = \pi \int_0^8 \left(\frac{1}{4}x\right)^2 \, dx = \frac{\pi}{16}\left[\frac{1}{3}x^3\right]_0^8 = \frac{32}{3}\pi$

15. $V = \pi \int_0^2 \left[8^2 - \left(4y\right)^2\right] dy = \pi \left[64y - \frac{16}{3}y^3\right]_0^2$

$= \pi \left(128 - \frac{128}{3}\right) = \frac{256}{3}\pi$

16. $V = \pi \int_0^2 \left(8 - 4y\right)^2 \, dy = \pi \left[64y - 32y^2 + \frac{16}{3}y^3\right]_0^2$

$= \pi \left(128 - 128 + \frac{128}{3}\right) = \frac{128}{3}\pi$

17. $V = \pi\int_0^8 \left[2^2 - \left(2 - \frac{1}{4}x\right)^2\right] dx = \pi\int_0^8 \left(x - \frac{1}{16}x^2\right) dx$
$= \pi\left[\frac{1}{2}x^2 - \frac{1}{48}x^3\right]_0^8 = \pi\left(32 - \frac{32}{3}\right) = \frac{64}{3}\pi$

18. $V = \pi\int_0^8 \left[\left(\sqrt[3]{x}\right)^2 - \left(\frac{1}{4}x\right)^2\right] = \pi\int_0^8 \left(x^{2/3} - \frac{1}{16}x^2\right) dx$
$= \pi\left[\frac{3}{5}x^{5/3} - \frac{1}{48}x^3\right]_0^8 = \pi\left(\frac{96}{5} - \frac{32}{3}\right) = \frac{128}{15}\pi$

19. $V = \pi\int_0^2 \left[(4y)^2 - \left(y^3\right)^2\right] dy = \pi\int_0^2 \left(16y^2 - y^6\right) dy$
$= \pi\left[\frac{16}{3}y^3 - \frac{1}{7}y^7\right]_0^2 = \pi\left(\frac{128}{3} - \frac{128}{7}\right) = \frac{512}{21}\pi$

20. $V = \pi\int_0^8 \left[\left(2 - \frac{1}{4}x\right)^2 - \left(2 - \sqrt[3]{x}\right)^2\right] dx$
$= \pi\int_0^8 \left(-x + \frac{1}{16}x^2 + 4x^{1/3} - x^{2/3}\right) dx$
$= \pi\left[-\frac{1}{2}x^2 + \frac{1}{48}x^3 + 3x^{4/3} - \frac{3}{5}x^{5/3}\right]_0^8$
$= \pi\left(-32 + \frac{32}{3} + 48 - \frac{96}{5}\right) = \frac{112}{15}\pi$

21. $V = \pi\int_0^2 \left[\left(8 - y^3\right)^2 - (8 - 4y)^2\right] dy$
$= \pi\int_0^2 \left(-16y^3 + y^6 + 64y - 16y^2\right) dy$
$= \pi\left[-4y^4 + \frac{1}{7}y^7 + 32y^2 - \frac{16}{3}y^3\right]_0^2$
$= \pi\left(-64 + \frac{128}{7} + 128 - \frac{128}{3}\right) = \frac{832}{21}\pi$

22. $V = \pi\int_0^8 \left(2^2 - x^{2/3}\right) dx = \pi\left[4x - \frac{3}{5}x^{5/3}\right]_0^8$
$= \pi\left(32 - \frac{96}{5}\right) = \frac{64}{5}\pi$

23. $V = \pi\int_0^2 \left(y^3\right)^2\, dy = \pi\left[\frac{1}{7}y^7\right]_0^2 = \frac{128}{7}\pi$

24. $V = \pi\int_0^8 \left(2 - \sqrt[3]{x}\right)^2\, dx = \pi\int_0^8 \left(4 - 4x^{1/3} + x^{2/3}\right) dx$
$= \pi\left[4x - 3x^{4/3} + \frac{3}{5}x^{5/3}\right]_0^8$
$= \pi\left(32 - 48 + \frac{96}{5}\right) = \frac{16}{5}\pi$

25. $V = \pi\int_0^2 \left[8^2 - \left(8 - y^3\right)^2\right] dy = \pi\int_0^2 \left(16y^3 - y^6\right) dy$
$= \pi\left[4y^4 - \frac{1}{7}y^7\right]_0^2 = \pi\left(64 - \frac{128}{7}\right) = \frac{320}{7}\pi$

26. $V = \pi\int_1^e \left[1^2 - (\ln x)^2\right] dx$

27. $V = \pi\int_0^2 \left[5^2 - \left(y^2 + 1\right)^2\right] dy = \pi\int_0^2 \left(24 - y^4 - 2y^2\right) dy$

28. $x - 1 = (x - 4)^2 + 1 \quad\Leftrightarrow\quad x^2 - 9x + 18 = 0 \quad\Leftrightarrow$
$x = 3$ or 6, so
$V = \pi\int_3^6 \left[\left[6 - (x - 4)^2\right]^2 - (8 - x)^2\right] dx$
$= \pi\int_3^6 \left(x^4 - 16x^3 + 83x^2 - 144x + 36\right) dx$

29. $V = \pi\int_0^{\pi/2} \left[1^2 - (1 - \cos x)^2\right] dx$
$= \pi\int_0^{\pi/2} \left(2\cos x - \cos^2 x\right) dx$

30. $V = \pi\int_0^{\pi/2} \left[(1 + \cos x)^2 - 1^2\right] dx$
$= \pi\int_0^{\pi/2} \left(2\cos x + \cos^2 x\right) dx$

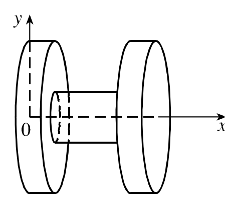
31. We see from the graph in Archived Problem 7.1.46 that the x-coordinates of the points of intersection are $x \approx -0.72$ and $x \approx 1.22$, with $\sqrt{x+1} > x^2$ on $[-0.72, 1.22]$, so the volume of revolution is about

$$\begin{aligned} &\pi\int_{-0.72}^{1.22} \left[\left(\sqrt{x+1}\right)^2 - \left(x^2\right)^2\right] dx \\ =\ &\pi\int_{-0.72}^{1.22} \left(x + 1 - x^4\right) dx \\ =\ &\pi\left[\tfrac{1}{2}x^2 + x - \tfrac{1}{5}x^5\right]_{-0.72}^{1.22} \\ \approx\ &5.80 \end{aligned}$$

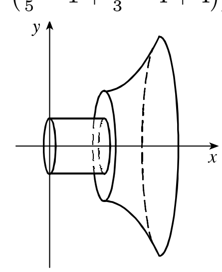
32. The x-coordinates of the points of intersection are $x = 0$ and $x \approx 1.17$, with $3x - x^3 > x^4$ on $[0, 1.17]$, so the volume of revolution is about

$$\begin{aligned} &\pi\int_0^{1.17} \left[\left(3x - x^3\right)^2 - \left(x^4\right)^2\right] dx \\ =\ &\pi\int_0^{1.17} \left[9x^2 - 6x^4 + x^6 - x^8\right] dx \\ =\ &\pi\left[3x^3 - \tfrac{6}{5}x^5 + \tfrac{1}{7}x^7 - \tfrac{1}{9}x^9\right]_0^{1.17} \\ \approx\ &6.74 \end{aligned}$$

33. $V = \pi\int_0^1 3^2\, dx + \pi\int_1^4 1^2\, dx + \pi\int_4^5 3^2\, dx$
$= 9\pi + 3\pi + 9\pi = 21\pi$



34. $V = \pi\int_0^1 \left(\frac{1}{2}\right)^2\, dx + \pi\int_1^2 \left(x^2 - 2x + 2\right)^2\, dx$
$= \frac{\pi}{4} + \pi\int_1^2 \left(x^4 - 4x^3 + 8x^2 - 8x + 4\right) dx$
$= \frac{\pi}{4} + \pi\left[\frac{1}{5}x^5 - x^4 + \frac{8}{3}x^3 - 4x^2 + 4x\right]_1^2$
$= \frac{\pi}{4} + \pi\left[\left(\frac{32}{5} - 16 + \frac{64}{3} - 16 + 8\right)\right.$
$\left. - \left(\frac{1}{5} - 1 + \frac{8}{3} - 4 + 4\right)\right] = \frac{127\pi}{60}$



35. The solid is obtained by rotating the region under the curve $y = \tan x$, from $x = 0$ to $x = \frac{\pi}{4}$, about the x-axis.

36. The solid is obtained by rotating the region bounded by the curve $x = y^3$ and the lines $y = 1$, $y = 2$, and $x = 0$ about the y-axis.

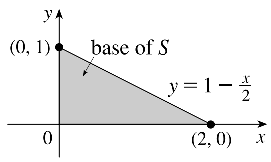
37. The solid is obtained by rotating the region between the curves $x = y$ and $x = \sqrt{y}$ about the y-axis.

38. The solid is obtained by rotating the region bounded by the curve $y = (x-2)^2$ and the line $y = 4$ about the x-axis.

39. The solid is obtained by rotating the region between the curves $y = 5 - 2x^2$ and $y = 5 - 2x$ about the x-axis. Or: The solid is obtained by rotating the region bounded by the curves $y = 2x$ and $y = 2x^2$ about the line $y = 5$.

40. The solid is obtained by rotating the region bounded by the curves $y = 2 + \cos x$ and $y = 2 + \sin x$ and the line $x = \frac{\pi}{2}$ about the x-axis.

41.



Since the area of a semicircle of diameter y is $\dfrac{\pi y^2}{8}$, we have

$$\begin{aligned} V &= \int_0^2 A(x)\,dx = \int_0^2 \tfrac{\pi}{8} y^2\,dx \\ &= \tfrac{\pi}{8}\int_0^2 \left(1 - \tfrac{1}{2}x\right)^2 dx = \tfrac{\pi}{4}\int_0^2 \left(\tfrac{1}{2}x - 1\right)^2 \tfrac{1}{2}\,dx \\ &= \frac{\pi}{4}\left[\tfrac{1}{3}\left(\tfrac{1}{2}x - 1\right)^3\right]_0^2 = \tfrac{\pi}{12}\left[0 - (-1)\right] = \tfrac{\pi}{12} \end{aligned}$$