9.2 CALCULUS WITH PARAMETRIC CURVES

A Click here for answers.

S Click here for solutions.

1–2 ▪ Find dy/dx.

1. $x = \sqrt{t} - t, \quad y = t^3 - t$

2. $x = t \ln t, \quad y = \sin^2 t$

3–6 ▪ Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

3. $x = t^2 + t, \ y = t^2 - t; \quad t = 0$

4. $x = t \sin t, \ y = t \cos t; \quad t = \pi$

5. $x = t^2 + t, \ y = \sqrt{t}; \quad t = 4$

6. $x = 2 \sin \theta, \ y = 3 \cos \theta; \quad \theta = \pi/4$

7–10 ▪ Find an equation of the tangent to the curve at the given point by two methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.

7. $x = 2t + 3, \ y = t^2 + 2t; \quad (5, 3)$

8. $x = 5 \cos t, \ y = 5 \sin t; \quad (3, 4)$

9. $x = 1 - t, \ y = 1 - t^2; \quad (1, 1)$

10. $x = t^3, \ y = t^2; \quad (1, 1)$

11–18 ▪ Find dy/dx and d^2y/dx^2.

11. $x = t^2 + t, \quad y = t^2 + 1$

12. $x = t + 2 \cos t, \quad y = \sin 2t$

13. $x = t^4 - 1, \quad y = t - t^2$

14. $x = t^3 + t^2 + 1, \quad y = 1 - t^2$

15. $x = \sin \pi t, \quad y = \cos \pi t$

16. $x = 1 + \tan t, \quad y = \cos 2t$

17. $x = e^{-t}, \quad y = te^{2t}$

18. $x = 1 + t^2, \quad y = t \ln t$

19. Estimate the area of the region enclosed by each loop of the curve

$$x = \sin t - 2 \cos t \qquad y = 1 + \sin t \cos t$$

20–23 ▪ Set up, but do not evaluate, an integral that represents the length of the curve.

20. $x = t^3, \quad y = t^4, \quad 0 \leqslant t \leqslant 1$

21. $x = t^2, \quad y = 1 + 4t, \quad 0 \leqslant t \leqslant 2$

22. $x = t \sin t, \quad y = t \cos t, \quad 0 \leqslant t \leqslant \pi/2$

23. $x = e^{-t}, \quad y = te^{2t}, \quad -1 \leqslant t \leqslant 1$

24–29 ▪ Find the length of the curve.

24. $x = t^3, \quad y = t^2, \quad 0 \leqslant t \leqslant 4$

25. $x = 3t - t^3, \quad y = 3t^2, \quad 0 \leqslant t \leqslant 2$

26. $x = 2 - 3 \sin^2 \theta, \quad y = \cos 2\theta, \quad 0 \leqslant \theta \leqslant \pi/2$

27. $x = 1 + 2 \sin \pi t, \quad y = 3 - 2 \cos \pi t, \quad 0 \leqslant t \leqslant 1$

28. $x = 5t^2 + 1, \quad y = 4 - 3t^2, \quad 0 \leqslant t \leqslant 2$

29. $x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \leqslant t \leqslant \pi$

CAS **30.** Graph the curve

$$x = t \cos t + \sin t \qquad y = t \sin t - \cos t \qquad -\pi \leqslant t \leqslant \pi$$

Then use a CAS or a table of integrals to find the exact length of the curve.

31. Use Simpson's Rule with $n = 10$ to estimate the length of the curve $x = \ln t$, $y = e^{-t}$, $1 \leqslant t \leqslant 2$.

32. Set up, but do not evaluate, an integral that represents the area of the surface obtained by rotating the curve $x = t^3$, $y = t^4$, $0 \leqslant t \leqslant 1$ about the x-axis.

33–35 ▪ Find the area of the surface obtained by rotating the given curve about the x-axis.

33. $x = t^2 + 2/t, \quad y = 8\sqrt{t}, \quad 1 \leqslant t \leqslant 9$

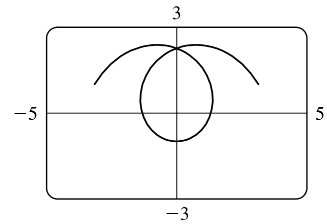
34. $x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \leqslant t \leqslant \pi/2$

35. $x = 2 \cos \theta - \cos 2\theta, \quad y = 2 \sin \theta - \sin 2\theta$

9.2 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $\dfrac{(3t^2-1)\,(2\sqrt{t})}{1-2\sqrt{t}}$
2. $\dfrac{2\sin t\cos t}{1+\ln t}$
3. $y=-x$
4. $y=\frac{1}{\pi}x-\pi$
5. $y=\frac{1}{36}x+\frac{13}{9}$
6. $y=-\frac{3}{2}x+3\sqrt{2}$
7. $y=2x-7$
8. $y=-\frac{3}{4}x+\frac{25}{4}$
9. $y=1$
10. $y=\frac{2}{3}x+\frac{1}{3}$
11. $1-\dfrac{1}{2t+1}, \dfrac{2}{(2t+1)^3}$
12. $\dfrac{2\cos 2t}{1-2\sin t}, \dfrac{4\,(\cos t-\sin 2t+\sin t\sin 2t)}{(1-2\sin t)^3}$
13. $\frac{1}{4}t^{-3}-\frac{1}{2}t^{-2}, \dfrac{-3+4t}{16t^7}$
14. $-\dfrac{2}{3t+2}, \dfrac{6}{t\,(3t+2)^3}$
15. $-\tan\pi t, -\sec^3\pi t$
16. $-4\sin t\cos^3 t, 4\cos^4 t\,(3\sin^2 t-\cos^2 t)$
17. $-(2t+1)\,e^{3t}, (6t+5)\,e^{4t}$
18. $\dfrac{1+\ln t}{2t}, -\dfrac{\ln t}{4t^3}$
19. $\frac{2\sqrt{5}}{5}$
20. $\int_0^1 t^2\sqrt{9+16t^2}\,dt$
21. $2\int_0^2\sqrt{t^2+4}\,dt$
22. $\int_0^{\pi/2}\sqrt{1+t^2}\,dt$
23. $\int_{-1}^1\sqrt{e^{-2t}+e^{4t}\,(1+2t)^2}\,dt$
24. $\frac{8}{27}\left(37^{3/2}-1\right)$
25. 14
26. $\sqrt{13}$
27. 2π
28. $4\sqrt{34}$
29. $\sqrt{2}\,(e^{\pi}-1)$
30. 

$\pi\sqrt{\pi^2+4}+4\ln\left(\pi+\sqrt{\pi^2+4}\right)-4\ln 2$

31. 0.7314
32. $\int_0^1 2\pi t^6\sqrt{9+16t^2}\,dt$
33. $\frac{47{,}104}{15}\pi$
34. $\frac{2\sqrt{2}\pi}{5}\,(2e^{\pi}-1)$
35. $\frac{128\pi}{5}$

9.2 SOLUTIONS

E Click here for exercises.

1. $x = \sqrt{t} - t,\ y = t^3 - t \quad\Rightarrow\quad \dfrac{dy}{dt} = 3t^2 - 1,$
$\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}} - 1$, and
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 1}{1/\left(2\sqrt{t}\right) - 1} = \frac{\left(3t^2 - 1\right)\left(2\sqrt{t}\right)}{1 - 2\sqrt{t}}$$

2. $x = t\ln t,\ y = \sin^2 t \quad\Rightarrow\quad \dfrac{dy}{dt} = 2\sin t\cos t,$
$\dfrac{dx}{dt} = t\left(\dfrac{1}{t}\right) + (\ln t)\cdot 1 = 1 + \ln t$, and
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\sin t\cos t}{1 + \ln t}$$

3. $x = t^2 + t,\ y = t^2 - t;\ t = 0.\ \dfrac{dy}{dt} = 2t - 1,\ \dfrac{dx}{dt} = 2t + 1,$
so $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t - 1}{2t + 1}$. When $t = 0$, $x = y = 0$
and $\dfrac{dy}{dx} = -1$. An equation of the tangent is
$y - 0 = (-1)(x - 0)$ or $y = -x$.

4. $x = t\sin t,\ y = t\cos t;\ t = \pi.\ \dfrac{dy}{dt} = \cos t - t\sin t,$
$\dfrac{dx}{dt} = \sin t + t\cos t$, and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\cos t - t\sin t}{\sin t + t\cos t}$.
When $t = \pi$, $(x, y) = (0, -\pi)$ and $\dfrac{dy}{dx} = \dfrac{-1}{-\pi} = \dfrac{1}{\pi}$, so an equation of the tangent is $y + \pi = \frac{1}{\pi}(x - 0)$ or
$y = \frac{1}{\pi}x - \pi$.

5. $x = t^2 + t,\ y = \sqrt{t};\ t = 4.\ \dfrac{dy}{dt} = \dfrac{1}{2\sqrt{t}},\ \dfrac{dx}{dt} = 2t + 1$, so
$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1}{2\sqrt{t}\,(2t + 1)}$. When $t = 4$,
$(x, y) = (20, 2)$ and $\dfrac{dy}{dx} = \dfrac{1}{36}$, so an equation of the tangent is $y - 2 = \frac{1}{36}(x - 20)$ or $y = \frac{1}{36}x + \frac{13}{9}$.

6. $x = 2\sin\theta,\ y = 3\cos\theta;\ \theta = \frac{\pi}{4}.\ \dfrac{dx}{d\theta} = 2\cos\theta,$
$\dfrac{dy}{d\theta} = -3\sin\theta,\ \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = -\frac{3}{2}\tan\theta$. When $\theta = \frac{\pi}{4}$,
$(x, y) = \left(\sqrt{2}, \frac{3\sqrt{2}}{2}\right)$, and $dy/dx = -\frac{3}{2}$, so an equation of the tangent is $y - \frac{3\sqrt{2}}{2} = -\frac{3}{2}\left(x - \sqrt{2}\right)$ or
$y = -\frac{3}{2}x + 3\sqrt{2}$.

7. (a) $x = 2t + 3,\ y = t^2 + 2t;\ (5, 3).\ \dfrac{dy}{dt} = 2t + 2,\ \dfrac{dx}{dt} = 2,$
and $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = t + 1$. At $(5, 3)$, $t = 1$ and $\dfrac{dy}{dx} = 2$,
so the tangent is $y - 3 = 2(x - 5)$ or $y = 2x - 7$.

(b) $y = t^2 + 2t = \left(\dfrac{x - 3}{2}\right)^2 + 2\left(\dfrac{x - 3}{2}\right)$
$= \dfrac{(x - 3)^2}{4} + x - 3$
so $\dfrac{dy}{dx} = \dfrac{x - 3}{2} + 1$. When $x = 5$, $\dfrac{dy}{dx} = 2$, so an equation of the tangent is $y = 2x - 7$, as before.

8. (a) $x = 5\cos t,\ y = 5\sin t;\ (3, 4).\ \dfrac{dy}{dt} = 5\cos t,$
$\dfrac{dx}{dt} = -5\sin t,\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = -\cot t$. At $(3, 4)$,
$t = \tan^{-1}\dfrac{y}{x} = \tan^{-1}\frac{4}{3}$, so $\dfrac{dy}{dx} = -\dfrac{3}{4}$, and an equation of the tangent is $y - 4 = -\frac{3}{4}(x - 3)$ or $y = -\frac{3}{4}x + \frac{25}{4}$.

(b) $x^2 + y^2 = 25$, so $2x + 2y\dfrac{dy}{dx} = 0$, or $\dfrac{dy}{dx} = -\dfrac{x}{y}$. At
$(3, 4)$, $\dfrac{dy}{dx} = -\dfrac{3}{4}$, and as in part (a), an equation of the tangent is $y = -\frac{3}{4}x + \frac{25}{4}$.

9. (a) $x = 1 - t,\ y = 1 - t^2;\ (1, 1).\ \dfrac{dy}{dt} = -2t,\ \dfrac{dx}{dt} = -1,$
$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = 2t$. At $(1, 1)$, $t = 0$, so $\dfrac{dy}{dx} = 0$, and the tangent is $y - 1 = 0(x - 1)$ or $y = 1$.

(b) $y = 1 - t^2 = 1 - \left(1 - x^2\right) = 2x - x^2$, so
$[dy/dx]_{x=1} = [2 - 2x]_{x-1} = 0$, and as in part (a), the tangent is $y = 1$.

10. (a) $x = t^3,\ y = t^2;\ (1, 1).\ \dfrac{dy}{dt} = 2t,\ \dfrac{dx}{dt} = 3t^2$, and
$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{3t^2} = \dfrac{2}{3t}$ (for $t \neq 0$). At $(1, 1)$, we have $t = 1$ and $dy/dx = \frac{2}{3}$, so an equation of the tangent is $y - 1 = \frac{2}{3}(x - 1)$ or $y = \frac{2}{3}x + \frac{1}{3}$.

(b) $y = x^{2/3}$, so $dy/dx = \frac{2}{3}x^{-1/3}$. When $x = 1$,
$dy/dx = \frac{2}{3}$, so the tangent is $y = \frac{2}{3}x + \frac{1}{3}$ as before.

11. $x = t^2 + t,\ y = t^2 + 1.$
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2t + 1} = 1 - \frac{1}{2t + 1};$$
$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{2}{(2t + 1)^2};$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d\,(dy/dx)\,/dt}{dx/dt} = \frac{2}{(2t + 1)^3}$$

12. $x = t + 2\cos t,\ y = \sin 2t.\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2\cos 2t}{1 - 2\sin t};$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{(1 - 2\sin t)(-4\sin 2t) - 2\cos 2t\,(-2\cos t)}{(1 - 2\sin t)^2} = \frac{4(\cos t - \sin 2t + \sin t \sin 2t)}{(1 - 2\sin t)^2};$$

$$\frac{d^2y}{dx^2} = \frac{d\,(dy/dx)/dt}{dx/dt} = \frac{4(\cos t - \sin 2t + \sin t \sin 2t)}{(1 - 2\sin t)^3}$$

13. $x = t^4 - 1,\ y = t - t^2 \quad\Rightarrow\quad \dfrac{dy}{dt} = 1 - 2t,\ \dfrac{dx}{dt} = 4t^3,$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 2t}{4t^3} = \tfrac{1}{4}t^{-3} - \tfrac{1}{2}t^{-2};$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -\tfrac{3}{4}t^{-4} + t^{-3},$$

$$\frac{d^2y}{dx^2} = \frac{d\,(dy/dx)/dt}{dx/dt} = \frac{-\frac{3}{4}t^{-4} + t^{-3}}{4t^3} \cdot \frac{4t^4}{4t^4} = \frac{-3 + 4t}{16t^7}.$$

14. $x = t^3 + t^2 + 1,\ y = 1 - t^2.\ \dfrac{dy}{dt} = -2t,$

$$\frac{dx}{dt} = 3t^2 + 2t;\ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2t}{3t^2 + 2t} = -\frac{2}{3t + 2};$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{6}{(3t + 2)^2};$$

$$\frac{d^2y}{dx^2} = \frac{d\,(dy/dx)/dt}{dx/dt} = \frac{6}{t\,(3t + 2)^3}.$$

15. $x = \sin \pi t,\ y = \cos \pi t.$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\pi \sin \pi t}{\pi \cos \pi t} = -\tan \pi t;$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d\,(dy/dx)/dt}{dx/dt} = \frac{-\pi \sec^2 \pi t}{\pi \cos \pi t} = -\sec^3 \pi t.$$

16. $x = 1 + \tan t,\ y = \cos 2t \quad\Rightarrow\quad \dfrac{dy}{dt} = -2\sin 2t,$

$$\frac{dx}{dt} = \sec^2 t,$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin 2t}{\sec^2 t} = -4\sin t \cos t \cdot \cos^2 t = -4\sin t \cos^3 t;$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -4\sin t\,(3\cos^2 t)(-\sin t) - 4\cos^4 t = 12\sin^2 t \cos^2 t - 4\cos^4 t,$$

$$\frac{d^2y}{dx^2} = \frac{d\,(dy/dx)/dt}{dx/dt} = \frac{4\cos^2 t\,(3\sin^2 t - \cos^2 t)}{\sec^2 t} = 4\cos^4 t\,(3\sin^2 t - \cos^2 t).$$

17. $x = e^{-t},\ y = te^{2t}.$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(2t + 1)e^{2t}}{-e^{-t}} = -(2t + 1)e^{3t};$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -3(2t + 1)e^{3t} - 2e^{3t} = -(6t + 5)e^{3t};$$

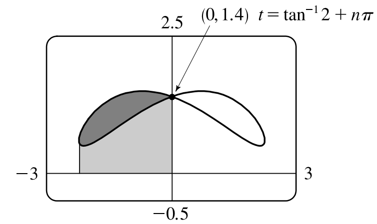
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d\,(dy/dx)/dt}{dx/dt} = \frac{-(6t + 5)e^{3t}}{-e^{-t}} = (6t + 5)e^{4t}.$$

18. $x = 1 + t^2,\ y = t\ln t.\ \dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1 + \ln t}{2t};$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{2t\,(1/t) - (1 + \ln t)\,2}{(2t)^2} = -\frac{\ln t}{2t^2};$$

$$\frac{d^2y}{dx^2} = \frac{d\,(dy/dx)/dt}{dx/dt} = -\frac{\ln t}{4t^3}.$$

19.



The graph of $x = \sin t - 2\cos t,\ y = 1 + \sin t \cos t$ is symmetric about the y-axis. The graph intersects the y-axis when $x = 0 \quad\Rightarrow\quad \sin t - 2\cos t = 0 \quad\Rightarrow\quad \sin t = 2\cos t \quad\Rightarrow\quad \tan t = 2 \quad\Rightarrow\quad t = \tan^{-1} 2 + n\pi$. The left loop is traced in a clockwise direction from $t = \tan^{-1} 2 - \pi$ to $t = \tan^{-1} 2$, so the area of the loop is given (as in Example 4) by

$$A = \int_{\tan^{-1} 2 - \pi}^{\tan^{-1} 2} y\,dx \approx \int_{-2.0344}^{1.1071} (1 + \sin t \cos t)(\cos t + 2\sin t)\,dt \approx 0.8944$$

This integral can be evaluated exactly; its value is $\frac{2}{5}\sqrt{5}$.

20. $L = \int_0^1 \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$

and $dx/dt = 3t^2,\ dy/dt = 4t^3 \quad\Rightarrow$

$L = \int_0^1 \sqrt{9t^4 + 16t^6}\,dt = \int_0^1 t^2\sqrt{9 + 16t^2}\,dt$

21. $L = \int_0^2 \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$ and $dx/dt = 2t$, $dy/dt = 4$, so $L = \int_0^2 \sqrt{4t^2 + 16}\,dt = 2\int_0^2 \sqrt{t^2 + 4}\,dt$.

22. $\dfrac{dx}{dt} = \sin t + t\cos t$ and $\dfrac{dy}{dt} = \cos t - t\sin t \quad\Rightarrow$

$$L = \int_0^{\pi/2} \sqrt{(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2}\,dt = \int_0^{\pi/2} \sqrt{1 + t^2}\,dt$$

23. $dx/dt = -e^{-t}$ and $dy/dt = e^{2t} + 2te^{2t} = e^{2t}(1 + 2t)$, so $L = \int_{-1}^{1} \sqrt{e^{-2t} + e^{4t}(1 + 2t)^2}\,dt$.

24. $x = t^3,\ y = t^2,\ 0 \le t \le 4.$

$(dx/dt)^2 + (dy/dt)^2 = (3t^2)^2 + (2t)^2 = 9t^4 + 4t^2.$

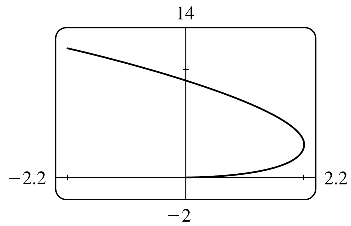
$$L = \int_0^4 \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = \int_0^4 \sqrt{9t^4 + 4t^2}\,dt = \int_0^4 t\sqrt{9t^2 + 4}\,dt$$

$$= \tfrac{1}{18}\int_4^{148} \sqrt{u}\,du \text{ (where } u = 9t^2 + 4) = \tfrac{1}{18}\left(\tfrac{2}{3}\right)\left[u^{3/2}\right]_4^{148} = \tfrac{1}{27}\left(148^{3/2} - 4^{3/2}\right) = \tfrac{8}{27}\left(37^{3/2} - 1\right)$$

25. $x = 3t - t^3,\ y = 3t^2,\ 0 \le t \le 2.$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(3 - 3t^2\right)^2 + (6t)^2 = 9\left(1 + 2t^2 + t^4\right) = \left[3\left(1+t^2\right)\right]^2$$

$$L = \int_0^2 3\left(1+t^2\right)dt = \left[3t + t^3\right]_0^2 = 14$$



26. $x = 2 - 3\sin^2\theta,\ y = \cos 2\theta,\ 0 \le \theta \le \frac{\pi}{2}.$

$$(dx/d\theta)^2 + (dy/d\theta)^2 = (-6\sin\theta\cos\theta)^2 + (-2\sin 2\theta)^2 = (-3\sin 2\theta)^2 + (-2\sin 2\theta)^2 = 13\sin^2 2\theta$$

$\Rightarrow$

$$L = \int_0^{\pi/2} \sqrt{13}\sin 2\theta d\theta = \left[-\tfrac{\sqrt{13}}{2}\cos 2\theta\right]_0^{\pi/2} = -\tfrac{\sqrt{13}}{2}(-1-1) = \sqrt{13}$$

27. $x = 1 + 2\sin\pi t,\ y = 3 - 2\cos\pi t,\ 0 \le t \le 1.$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (2\pi\cos\pi t)^2 + (2\pi\sin\pi t)^2 = 4\pi^2$$

$$\Rightarrow\quad L = \int_0^1 \sqrt{(dx/dt)^2 + (dy/dt^2)}\,dt = \int_0^1 2\pi\,dt = 2\pi$$

28. $x = 5t^2 + 1,\ y = 4 - 3t^2,\ 0 \le t \le 2.$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (10t)^2 + (-6t)^2 = 136t^2 \quad\Rightarrow$$

$$L = \int_0^2 \sqrt{136t^2}\,dt = \int_0^2 \sqrt{136}\,t\,dt = \left[\tfrac{1}{2}\cdot 2\sqrt{34}\,t^2\right]_0^2 = 4\sqrt{34}$$

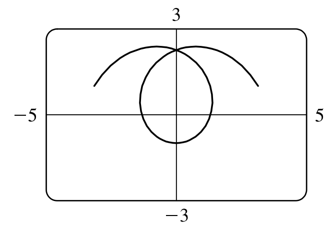
29. $x = e^t\cos t,\ y = e^t\sin t,\ 0 \le t \le \pi.$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left[e^t(\cos t - \sin t)\right]^2 + \left[e^t(\cos t + \sin t)\right]^2 = e^{2t}\left(2\cos^2 t + 2\sin^2 t\right) = 2e^{2t}$$

$$\Rightarrow\quad L = \int_0^\pi \sqrt{2}\,e^t\,dt = \sqrt{2}\left(e^\pi - 1\right)$$

30. $x = t\cos t + \sin t,\ y = t\sin t - \cos t,\ -\pi \le t \le \pi.$
$dx/dt = -t\sin t + 2\cos t$ and $dy/dt = t\cos t + 2\sin t$, so $(dx/dt)^2 + (dy/dt)^2 = t^2\sin^2 t - 4t\sin t\cos t + 4\cos^2 t + t^2\cos^2 t + 4t\sin t\cos t + 4\sin^2 t = t^2 + 4$ and

$$L = \int_{-\pi}^{\pi}\sqrt{t^2+4}\,dt = 2\int_0^\pi \sqrt{t^2+4}\,dt \stackrel{21}{=} 2\left[\tfrac{1}{2}t\sqrt{t^2+4} + 2\ln\left(t + \sqrt{t^2+4}\right)\right]_0^\pi$$
$$= 2\left[\tfrac{\pi}{2}\sqrt{\pi^2+4} + 2\ln\left(\pi + \sqrt{\pi^2+4}\right) - 2\ln 2\right] = \pi\sqrt{\pi^2+4} + 4\ln\left(\pi + \sqrt{\pi^2+4}\right) - 4\ln 2 \approx 16.633506$$



31. $x = \ln t$ and $y = e^{-t} \quad\Rightarrow\quad \dfrac{dx}{dt} = \dfrac{1}{t}$ and $\dfrac{dy}{dt} = -e^{-t} \quad\Rightarrow$
$L = \int_1^2 \sqrt{t^{-2} + e^{-2t}}\,dt$. Using Simpson's Rule with $n = 10$, $\Delta x = (2-1)/10 = 0.1$ and $f(t) = \sqrt{t^{-2} + e^{-2t}}$ we get $L \approx \frac{0.1}{3}\left[f(1.0) + 4f(1.1) + 2f(1.2) + \cdots + 2f(1.8) + 4f(1.9) + f(2.0)\right] \approx 0.7314$

32. $x = t^3$ and $y = t^4 \quad\Rightarrow\quad dx/dt = 3t^2$ and $dy/dt = 4t^3$.
So $S = \int_0^1 2\pi t^4\sqrt{9t^4 + 16t^6}\,dt = \int_0^1 2\pi t^6\sqrt{9 + 16t^2}\,dt$.

33.
$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(2t - \frac{2}{t^2}\right)^2 + \left(\frac{4}{\sqrt{t}}\right)^2 = 4t^2 + \frac{8}{t} + \frac{4}{t^4} = 4\left(t + \frac{1}{t^2}\right)^2$$

$$S = \int_1^9 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 2\pi\int_1^9 \left(8\sqrt{t}\right)2\left(t + \frac{1}{t^2}\right)dt$$
$$= 32\pi\int_1^9 \left(t^{3/2} + t^{-3/2}\right)dt = 32\pi\left[\tfrac{2}{5}t^{5/2} - 2t^{-1/2}\right]_1^9$$
$$= 32\pi\left\{\left[\tfrac{2}{5}(243) - 2\left(\tfrac{1}{3}\right)\right] - \left[\tfrac{2}{5}(1) - 2(1)\right]\right\} = \tfrac{47{,}104}{15}\pi$$

34. $x = e^t\cos t,\ y = e^t\sin t,\ 0 \le t \le \frac{\pi}{2}.$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left[e^t(\cos t - \sin t)\right]^2 + \left[e^t(\cos t + \sin t)\right]^2 = e^{2t}\left(2\cos^2 t + 2\sin^2 t\right) = 2e^{2t}, \text{ so}$$

$$S = \int_0^{\pi/2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{\pi/2} 2\pi e^t\sin t\sqrt{2}e^t\,dt \stackrel{98}{=} 2\sqrt{2}\pi\int_0^{\pi/2} e^{2t}\sin t\,dt$$
$$= \left[2\sqrt{2}\pi\frac{e^{2t}}{5}(2\sin t - \cos t)\right]_0^{\pi/2} = \tfrac{2\sqrt{2}}{5}\pi\left[2e^\pi - (-1)\right] = \tfrac{2\sqrt{2}\pi}{5}\left(2e^\pi - 1\right)$$

35. $\left(\dfrac{dx}{d\theta}\right)^2+\left(\dfrac{dy}{d\theta}\right)^2$

$$\begin{aligned}
&= (-2\sin\theta+2\sin 2\theta)^2+(2\cos\theta-2\cos 2\theta)^2\\
&= 4\left[\left(\sin^2\theta-2\sin\theta\sin 2\theta+\sin^2 2\theta\right)\right.\\
&\qquad\qquad\left.+\left(\cos^2\theta-2\cos\theta\cos 2\theta+\cos^2 2\theta\right)\right]\\
&= 4\left[1+1-2\left(\cos 2\theta\cos\theta+\sin 2\theta\sin\theta\right)\right]\\
&= 8\left[1-\cos\left(2\theta-\theta\right)\right]=8\left(1-\cos\theta\right)
\end{aligned}$$

Note that $x\left(2\pi-\theta\right)=x\left(\theta\right)$ and $y\left(2\pi-\theta\right)=-y\left(\theta\right)$, so the piece of the curve from $\theta=0$ to $\theta=\pi$ generates the same surface as the piece from $\theta=\pi$ to $\theta=2\pi$. Note also that $y=2\sin\theta-\sin 2\theta=2\sin\theta\left(1-\cos\theta\right)$. So

$$\begin{aligned}
S &= \int_0^{\pi} 2\pi\cdot 2\sin\theta\left(1-\cos\theta\right)2\sqrt{2}\sqrt{1-\cos\theta}\,d\theta\\
&= 8\sqrt{2}\pi\int_0^{\pi}\left(1-\cos\theta\right)^{3/2}\sin\theta\,d\theta\\
&= 8\sqrt{2}\pi\int_0^{2}\sqrt{u^3}\,du\ \left(u=1-\cos\theta,\ du=\sin\theta\,d\theta\right)\\
&= \left[8\sqrt{2}\pi\left(\tfrac{2}{5}u^{5/2}\right)\right]_0^2\\
&= \frac{128\pi}{5}
\end{aligned}$$