11.2 LIMITS AND CONTINUITY

A Click here for answers.

S Click here for solutions.

1–22 ▪ Find the limit, if it exists, or show that the limit does not exist.

1. $\lim_{(x,y)\to(2,3)} (x^2y^2 - 2xy^5 + 3y)$

2. $\lim_{(x,y)\to(-3,4)} (x^3 + 3x^2y^2 - 5y^3 + 1)$

3. $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^3 + x^3y^2 - 5}{2 - xy}$

4. $\lim_{(x,y)\to(-2,1)} \dfrac{x^2 + xy + y^2}{x^2 - y^2}$

5. $\lim_{(x,y)\to(\pi,\pi)} x \sin\left(\dfrac{x + y}{4}\right)$

6. $\lim_{(x,y)\to(1,4)} e^{\sqrt{x + 2y}}$

7. $\lim_{(x,y)\to(0,0)} \dfrac{x - y}{x^2 + y^2}$

8. $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2 + 2y^2}$

9. $\lim_{(x,y)\to(0,0)} \dfrac{(x + y)^2}{x^2 + y^2}$

10. $\lim_{(x,y)\to(0,0)} \dfrac{8x^2y^2}{x^4 + y^4}$

11. $\lim_{(x,y)\to(0,0)} \dfrac{x^3 + xy^2}{x^2 + y^2}$

12. $\lim_{(x,y)\to(0,0)} \dfrac{xy + 1}{x^2 + y^2 + 1}$

13. $\lim_{(x,y)\to(0,0)} \dfrac{x^3y^2}{x^2 + y^2}$

14. $\lim_{(x,y)\to(2,0)} \dfrac{xy - 2y}{x^2 + y^2 - 4x + 4}$

15. $\lim_{(x,y)\to(0,0)} \dfrac{\sqrt{x^2y^2 + 1} - 1}{x^2 + y^2}$

16. $\lim_{(x,y)\to(0,1)} \dfrac{xy - x}{x^2 + y^2 - 2y + 1}$

17. $\lim_{(x,y)\to(1,-1)} \dfrac{x^2 + y^2 - 2x - 2y}{x^2 + y^2 - 2x + 2y + 2}$

18. $\lim_{(x,y,z)\to(1,2,3)} \dfrac{xz^2 - y^2z}{xyz - 1}$

19. $\lim_{(x,y,z)\to(2,3,0)} [xe^z + \ln(2x - y)]$

20. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x^2 - y^2 - z^2}{x^2 + y^2 + z^2}$

21. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{xy + yz + zx}{x^2 + y^2 + z^2}$

22. $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x^2y^2z^2}{x^2 + y^2 + z^2}$

23–24 ▪ Find $h(x, y) = g(f(x, y))$ and the set on which h is continuous.

23. $g(t) = e^{-t} \cos t, \quad f(x, y) = x^4 + x^2y^2 + y^4$

24. $g(z) = \sin z, \quad f(x, y) = y \ln x$

25–38 ▪ Determine the set of points at which the function is continuous.

25. $F(x, y) = \dfrac{x^2 + y^2 + 1}{x^2 + y^2 - 1}$

26. $F(x, y) = \dfrac{x^6 + x^3y^3 + y^6}{x^3 + y^3}$

27. $F(x, y) = \tan(x^4 - y^4)$

28. $G(x, y) = e^{xy} \sin(x + y)$

29. $F(x, y) = \dfrac{1}{x^2 - y}$

30. $F(x, y) = \ln(2x + 3y)$

31. $G(x, y) = \sqrt{x + y} - \sqrt{x - y}$

32. $f(x, y, z) = \dfrac{xyz}{x^2 + y^2 - z}$

33. $G(x, y) = 2^{x \tan y}$

34. $f(x, y, z) = x \ln(yz)$

35. $f(x, y, z) = x + y\sqrt{x + z}$

36. $f(x, y) = \begin{cases} \dfrac{2x^2 - y^2}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

37. $f(x, y) = \begin{cases} \dfrac{x^2y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

38. $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + xy + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

39. Prove, using Definition 1, that

(a) $\lim_{(x,y)\to(a,b)} x = a$ (b) $\lim_{(x,y)\to(a,b)} y = b$

(c) $\lim_{(x,y)\to(a,b)} c = c$

40. Use polar coordinates to find

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

[If (r, θ) are polar coordinates of the point (x, y) with $r \geqslant o$, note that $r \to 0^+$ as $(x, y) \to (0, 0)$.]

11.2 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. -927

2. 86

3. $-\frac{5}{2}$

4. 1

5. π

6. e^3

7. Does not exist

8. Does not exist

9. Does not exist

10. Does not exist

11. 0

12. 1

13. 0

14. Does not exist

15. 0

16. Does not exist

17. Does not exist

18. $-\frac{3}{5}$

19. 2

20. Does not exist

21. Does not exist

22. 0

23. $h(x,y) = e^{-\left(x^4+x^2y^2+y^4\right)} \cos\left(x^4+x^2y^2+y^4\right)$, $\mathbb{R}^2$

24. $h(x,y) = \sin(y \ln x)$, $\{(x,y) \mid x > 0\}$

25. $\left\{(x,y) \mid x^2+y^2-1 \neq 0\right\}$

26. $\{(x,y) \mid y \neq -x\}$

27. $\left\{(x,y) \mid x^4 - y^4 \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\right\}$

28. $\mathbb{R}^2$

29. $\left\{(x,y) \mid y \neq x^2\right\}$

30. $\{(x,y) \mid 2x+3y > 0\}$

31. $\{(x,y) \mid |y| \leq x\}$

32. $\left\{(x,y,z) \mid z \neq x^2+y^2\right\}$

33. $\left\{(x,y) \mid y \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\right\}$

34. $\{(x,y,z) \mid yz > 0\}$

35. $\{(x,y,z) \mid x+z \geq 0\}$

36. $\{(x,y) \mid (x,y) \neq (0,0)\}$

37. $\mathbb{R}^2$

38. $\{(x,y) \mid (x,y) \neq (0,0)\}$

40. 1

11.2 SOLUTIONS

E Click here for exercises.

1. The function is a polynomial, so the limit equals $(2^2)(3^2) - 2(2)(3^5) + 3(3) = -927$.

2. The function is a polynomial, so the limit equals $(-3)^3 + 3(-3)^2(4)^2 - 5(4)^3 + 1 = 86$.

3. Since this is a rational function defined at $(0,0)$, the limit equals $(0+0-5)/(2-0) = -\frac{5}{2}$.

4. This is a rational function defined at $(-2,1)$, so the limit equals $(4-2+1)/(4-1) = 1$.

5. The product of two functions continuous at (π,π), so the limit equals $\pi \sin \frac{\pi+\pi}{4} = \pi$.

6. The composition of two continuous functions, so the limit equals $e^{\sqrt{1+8}} = e^3$.

7. Let $f(x,y) = \dfrac{x-y}{x^2+y^2}$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = \dfrac{x}{x^2} = \dfrac{1}{x}$ and $\lim_{x\to 0} f(x,0)$ doesn't exist. Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist.

8. Let $f(x,y) = \dfrac{2xy}{x^2+2y^2}$. As $(x,y)\to(0,0)$ along the x-axis, $f(x,y)\to 0$. But as $(x,y)\to(0,0)$ along the line $y=x$, $f(x,x) = \dfrac{2x^2}{3x^2}$, so $f(x,y)\to\frac{2}{3}$ as $(x,y)\to(0,0)$ along this line. So the limit doesn't exist.

9. $f(x,y) = \dfrac{(x+y)^2}{x^2+y^2}$. As $(x,y)\to(0,0)$ along the x-axis, $f(x,y)\to 1$. But as $(x,y)\to(0,0)$ along the line $y=x$, $f(x,x) = \dfrac{4x^2}{2x^2} = 2$ for $x\neq 0$, so $f(x,y)\to 2$. Thus, the limit does not exist.

10. $f(x,y) = \dfrac{8x^2y^2}{x^4+y^4}$. Approaching $(0,0)$ along the x-axis gives $f(x,y)\to 0$. Approaching $(0,0)$ along the line $y=x$, $f(x,x) = \dfrac{8x^4}{2x^4} = 4$ for $x\neq 0$, so along this line $f(x,y)\to 4$ as $(x,y)\to(0,0)$. Thus the limit doesn't exist.

11. $$\lim_{(x,y)\to(0,0)} \frac{x^3+xy^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x = 0$$

12. Since $\dfrac{xy+1}{x^2+y^2+1}$ is a rational function defined at $(0,0)$ the limit is $\dfrac{0+1}{0+0+1} = 1$.

13. $f(x,y) = \dfrac{x^3y^2}{x^2+y^2}$. We use the Squeeze Theorem: $0 \le \dfrac{|x^3y^2|}{x^2+y^2} \le |x^3|$ since $y^2 \le x^2+y^2$, and $|x^3|\to 0$ as $(x,y)\to(0,0)$. So $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

14. $f(x,y) = \dfrac{xy-2y}{x^2+y^2-4x+4} = \dfrac{y(x-2)}{y^2+(x-2)^2}$. Then $f(x,0) = 0$ for $x\neq 2$, so $f(x,y)\to 0$ as $(x,y)\to(2,0)$ along the x-axis. But $f(x,x-2) = \dfrac{(x-2)(x-2)}{(x-2)^2+(x-2)^2} = \dfrac{(x-2)^2}{2(x-2)^2} = \dfrac{1}{2}$ for $x\neq 2$, so $f(x,y)\to\frac{1}{2}$ as $(x,y)\to(2,0)$ along the line $y = x-2$ $(x\neq 2)$. Thus, the limit doesn't exist.

15. We have
$$0 \le \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = \frac{x^2y^2}{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)} \text{ (rationalize)}$$
$$\le \frac{x^2y^2}{2(x^2+y^2)} \le x^2 \text{ [since } y^2 \le 2(x^2+y^2)\text{]}$$
But $\lim_{(x,y)\to(0,0)} x^2 = 0$, so, by the Squeeze Theorem,
$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = 0.$$

16. Let $f(x,y) = \dfrac{xy-x}{x^2+y^2-2y+1}$. Then $f(0,y) = 0$ for $y\neq 1$, so $f(x,y)\to 0$ as $(x,y)\to(0,1)$ along the y-axis. But $f(x,x+1) = \dfrac{x(x+1-1)}{x^2+(x+1-1)^2} = \dfrac{1}{2}$ for $x\neq 0$ so $f(x,y)\to\frac{1}{2}$ as $(x,y)\to(0,1)$ along the line $y=x+1$. Thus, the limit doesn't exist.

17. Let
$$f(x,y) = \frac{x^2+y^2-2x-2y}{x^2+y^2-2x+2y+2} = \frac{(x-1)^2+(y-1)^2-2}{(x-1)^2+(y+1)^2}$$
Then $f(1,y) = \dfrac{(y-1)^2-2}{(y+1)^2}$. Thus, as $(x,y)\to(1,-1)$ along the line $x=1$, the limit of $f(x,y)$ doesn't exist and so the limit doesn't exist.

18. $\lim_{(x,y,z)\to(1,2,3)} \dfrac{xz^2-y^2z}{xyz-1} = \dfrac{1\cdot 3^2 - 2^2\cdot 3}{1\cdot 2\cdot 3-1} = -\dfrac{3}{5}$ since the function is continuous at $(1,2,3)$.

19. $$\lim_{(x,y,z)\to(2,3,0)} \left[xe^z + \ln(2x-y)\right] = (2)(e^0) + \ln(4-3) = 2$$
since the function is continuous at $(2,3,0)$.

20. Let $f(x,y,z) = \dfrac{x^2-y^2-z^2}{x^2+y^2+z^2}$. Then $f(x,0,0) = 1$ for $x\neq 0$ and $f(0,y,0) = -1$ for $y\neq 0$, so as $(x,y,z)\to(0,0,0)$ along the x-axis, $f(x,y,z)\to 1$ but as $(x,y,z)\to(0,0,0)$ along the y-axis, $f(x,y,z)\to -1$. Thus the limit doesn't exist.

21. Let $f(x,y,z) = \dfrac{xy+yz+zx}{x^2+y^2+z^2}$. Then $f(x,0,0) = 0$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the x-axis, $f(x,y,z) \to 0$. But $f(x,x,0) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the line $y = x, z = 0$, $f(x,y,z) \to \frac{1}{2}$. Thus the limit doesn't exist.

22. We can show that the limit along any line through the origin is 0 and thus suspect that this limit exists and equals 0. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $\left|\dfrac{x^2y^2z^2}{x^2+y^2+z^2} - 0\right| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2+z^2} < \delta$ or equivalently $\dfrac{x^2y^2z^2}{x^2+y^2+z^2} < \varepsilon$ whenever $0 < \sqrt{x^2+y^2+z^2} < \delta$. But $x^2 \leq x^2+y^2+z^2$ and similarly for y^2 and z^2, so $\dfrac{x^2y^2z^2}{x^2+y^2+z^2} \leq \dfrac{(x^2+y^2+z^2)^3}{x^2+y^2+z^2} = (x^2+y^2+z^2)^2$ for $x^2+y^2+z^2 \neq 0$. Thus choose $\delta = \varepsilon^{1/4}$ and let $0 < \sqrt{x^2+y^2+z^2} < \delta$. Then

$$\left|\frac{x^2y^2z^2}{x^2+y^2+z^2} - 0\right| \leq (x^2+y^2+z^2)^2 = \left(\sqrt{x^2+y^2+z^2}\right)^4 < \delta^4 = \left(\varepsilon^{1/4}\right)^4 = \varepsilon$$

Hence by definition $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2+y^2+z^2} = 0$.

Or: Use the Squeeze Theorem: $0 \leq \dfrac{x^2y^2z^2}{x^2+y^2+z^2} \leq x^2y^2$ since $z^2 \leq x^2+y^2+z^2$, and $x^2y^2 \to 0$ as $(x,y,z) \to (0,0,0)$.

23. $h(x,y) = g(f(x,y)) = g(x^4+x^2y^2+y^4)$

$$= e^{-(x^4+x^2y^2+y^4)} \cos(x^4+x^2y^2+y^4)$$

Since f is a polynomial it is continuous throughout $\mathbb{R}^2$ and g is the product of two functions, both of which are continuous on $\mathbb{R}$, h is continuous on $\mathbb{R}^2$.

24. $h(x,y) = g(f(x,y)) = \sin(y \ln x)$. Since $f(x,y) = y \ln x$ it is continuous on its domain $\{(x,y) \mid x > 0\}$ and g is continuous throughout $\mathbb{R}$. Thus h is continuous on its domain $D = \{(x,y) \mid x > 0\}$, the right half-plane excluding the y-axis.

25. $F(x,y)$ is a rational function and thus is continuous on its domain $D = \{(x,y) \mid x^2+y^2-1 \neq 0\}$, that is, F is continuous except on the circle $x^2+y^2 = 1$.

26. $F(x,y)$ is a rational function and thus is continuous on its domain $D = \{(x,y) \mid x^3+y^3 \neq 0\} = \{(x,y) \mid y \neq -x\}$, $\mathbb{R}^2$ except the line $y = -x$.

27. $F(x,y) = g(f(x,y))$ where $f(x,y) = x^4 - y^4$, a polynomial so continuous on $\mathbb{R}^2$ and $g(t) = \tan t$, continuous on its domain $\{t \mid t \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\}$. Thus F is continuous on its domain $D = \{(x,y) \mid x^4 - y^4 \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\}$.

28. $G(x,y) = g(x,y)\,f(x,y)$ where $g(x,y) = e^{xy}$ and $f(x,y) = \sin(x+y)$ both of which are continuous on $\mathbb{R}^2$. Thus G is continuous on $\mathbb{R}^2$.

29. $F(x,y) = \dfrac{1}{x^2-y}$ is a rational function and thus is continuous on its domain $\{(x,y) \mid x^2 - y \neq 0\} = \{(x,y) \mid y \neq x^2\}$, so F is continuous on $\mathbb{R}^2$ except the parabola $y = x^2$.

30. $F(x,y) = \ln(2x+3y) = g(f(x,y))$ where $f(x,y) = 2x+3y$, continuous on $\mathbb{R}^2$ and $g(t) = \ln t$, continuous on its domain $\{t \mid t > 0\}$. Thus F is continuous on its domain $D = \{(x,y) \mid 2x+3y > 0\}$.

31. $G(x,y) = g_1(f_1(x,y)) - g_2(f_2(x,y))$ where $f_1(x,y) = x+y$ and $f_2(x,y) = x-y$, both of which are polynomials so continuous on $\mathbb{R}^2$, and $g_1(t) = \sqrt{t}$, $g_2(s) = \sqrt{s}$, both of which are continuous on their respective domains $\{t \mid t \geq 0\}$ and $\{s \mid s \geq 0\}$. Thus $g_1 \circ f_1$ is continuous on its domain $D_1 = \{(x,y) \mid x+y \geq 0\} = \{(x,y) \mid y \geq -x\}$ and $g_2 \circ f_2$ is continuous on its domain $D_2 = \{(x,y) \mid x-y \geq 0\} = \{(x,y) \mid y \leq x\}$. Then G, being the difference of these two composite functions, is continuous on its domain

$$D = D_1 \cap D_2 = \{(x,y) \mid -x \leq y \leq x\}$$
$$= \{(x,y) \mid |y| \leq x\}$$

32. $f(x,y,z) = \dfrac{xyz}{x^2+y^2-z}$ is a rational function and thus is continuous on its domain $\{(x,y,z) \mid x^2+y^2-z \neq 0\} = \{(x,y,z) \mid z \neq x^2+y^2\}$, so f is continuous on $\mathbb{R}^3$ except on the circular paraboloid $z = x^2+y^2$.

33. $G(x,y) = g(f(x,y))$ where $f(x,y) = x \tan y$ which is continuous on its domain $\{(x,y) \mid y \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\}$ and $g(t) = 2^t$ which is continuous on $\mathbb{R}$. Thus $G(x,y)$ is continuous on its domain $D = \{(x,y) \mid y \neq (2n+1)\frac{\pi}{2}, n \text{ an integer}\}$.

34. $f(x,y,z) = xg(f(y,z))$ where $f(y,z) = yz$, continuous on $\mathbb{R}^2$ and $g(t) = \ln t$, continuous on its domain $\{t \mid t > 0\}$. Since $h(x) = x$ is continuous on $\mathbb{R}$, $f(x,y,z)$ is continuous on its domain $D = \{(x,y,z) \mid yz > 0\}$.

35. $f(x,y,z) = h(x) + k(y)\,g(f(x,z))$ where $h(x) = x$ and $k(y) = y$, both continuous on $\mathbb{R}$ and $f(x,z) = x+z$, continuous on $\mathbb{R}^2$, $g(t) = \sqrt{t}$ continuous on its domain $D = \{t \mid t \geq 0\}$. Thus f is continuous on its domain $D = \{(x,y,z) \mid x+z \geq 0\}$.

In Problems 36–38, each f is a piecewise defined function whose first piece is a rational function defined everywhere except at the origin. Thus each f is continuous on $\mathbb{R}^2$ except possibly at the origin. So for each we need only check $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

36. Letting $z = \sqrt{2}x$,
$\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2 - y^2}{2x^2 + y^2} = \lim\limits_{(z,y)\to(0,0)} \dfrac{z^2 - y^2}{z^2 + y^2}$ which doesn't exist by Example 1. Thus f is not continuous at $(0,0)$ and the largest set on which f is continuous is $\{(x,y) \mid (x,y) \neq (0,0)\}$.

37. Since $x^2 \leq 2x^2 + y^2$, we have $\left|\dfrac{x^2y^3}{2x^2+y^2}\right| \leq |y^3|$. We know that $|y^3| \to 0$ as $(x,y) \to (0,0)$. So, by the Squeeze Theorem, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^3}{2x^2+y^2} = 0$. Also $f(0,0) = 0$, so f is continuous at $(0,0)$. For $(x,y) \neq (0,0)$, $f(x,y)$ is equal to a rational function and is therefore continuous. Thus f is continuous throughout $\mathbb{R}^2$.

38. Let $g(x,y) = \dfrac{xy}{x^2+xy+y^2}$. Then $g(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $g(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. But $g(x,x) = \dfrac{x^2}{3x^2} = \frac{1}{3}$ for $x \neq 0$, so $g(x,y) \to \frac{1}{3}$ as $(x,y) \to (0,0)$ along the line $y = x$. Thus $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+xy+y^2}$ doesn't exist, so f is not continuous at $(0,0)$ and the largest set on which f is continuous is $\{(x,y) \mid (x,y) \neq (0,0)\}$.

39. (a) Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|x-a| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. But $|x-a| = \sqrt{(x-a)^2} \leq \sqrt{(x-a)^2+(y-b)^2}$. Thus setting $\delta = \varepsilon$ and letting $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$, we have $|x-a| \leq \sqrt{(x-a)^2+(y-b)^2} < \delta = \varepsilon$. Hence, by Definition 1, $\lim\limits_{(x,y)\to(a,b)} x = a$.

(b) The argument is the same as in (a) with the roles of x and y interchanged.

(c) Let $\varepsilon > 0$ be given and set $\delta = \varepsilon$. Then

$$|f(x,y) - L| = |c - c| = 0 \leq \sqrt{(x-a)^2+(y-b)^2} < \delta = \varepsilon$$

whenever $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$. Thus, by Definition 1, $\lim\limits_{(x,y)\to(a,b)} c = c$.

40. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2} = \lim\limits_{r\to0^+} \dfrac{\sin(r^2)}{r^2}$, which is an indeterminate form of type $\frac{0}{0}$. Using l'Hospital's Rule, we get

$$\lim_{r\to0^+} \frac{\sin(r^2)}{r^2} = \lim_{r\to0^+} \frac{2r\cos(r^2)}{2r} = \lim_{r\to0^+} \cos(r^2) = 1$$

Or: Use the fact that $\lim\limits_{\theta\to0} \dfrac{\sin\theta}{\theta} = 1$.