11.7 MAXIMUM AND MINIMUM VALUES

A Click here for answers.

S Click here for solutions.

1–9 ▪ Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

1. $f(x, y) = x^2 + y^2 + 4x - 6y$

2. $f(x, y) = 4x^2 + y^2 - 4x + 2y$

3. $f(x, y) = 2x^2 + y^2 + 2xy + 2x + 2y$

4. $f(x, y) = x^3 - 3xy + y^3$

5. $f(x, y) = x^2 + y^2 + x^2y + 4$

6. $f(x, y) = xy - 2x - y$

7. $f(x, y) = y\sqrt{x} - y^2 - x + 6y$

8. $f(x, y) = \dfrac{x^2y^2 - 8x + y}{xy}$

9. $f(x, y) = \dfrac{(x + y + 1)^2}{x^2 + y^2 + 1}$

10–14 ▪ Find the absolute maximum and minimum values of f on the set D.

10. $f(x, y) = 5 - 3x + 4y$,
D is the closed triangular region with vertices $(0, 0)$, $(4, 0)$, and $(4, 5)$

11. $f(x, y) = x^2 + 2xy + 3y^2$,
D is the closed triangular region with vertices $(-1, 1)$, $(2, 1)$, and $(-1, -2)$

12. $f(x, y) = y\sqrt{x} - y^2 - x + 6y$,
$D = \{(x, y) \mid 0 \leqslant x \leqslant 9, 0 \leqslant y \leqslant 5\}$

13. $f(x, y) = 1 + xy - x - y$,
D is the region bounded by the parabola $y = x^2$ and the line $y = 4$

14. $f(x, y) = 2x^2 + x + y^2 - 2$,
$D = \{(x, y) \mid x^2 + y^2 \leqslant 4\}$

15. Find the shortest distance from the point $(2, -2, 3)$ to the plane $6x + 4y - 3z = 2$.

16. Find the point on the plane $2x - y + z = 1$ that is closest to the point $(-4, 1, 3)$.

17. Find the point on the plane $x + 2y + 3z = 4$ that is closest to the origin.

18. Find the shortest distance from the point (x_0, y_0, z_0) to the plane $Ax + By + Cz + D = 0$.

11.7 ANSWERS

E Click here for exercises.

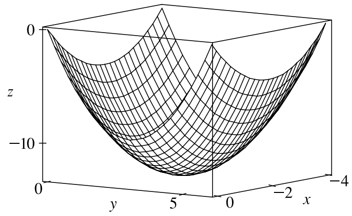
S Click here for solutions.

1. Minimum $f(-2,3) = -13$
2. Minimum $f\left(\frac{1}{2},-1\right) = -2$
3. Minimum $f(0,-1) = -1$
4. Minimum $f(1,1) = -1$, saddle point $(0,0)$
5. Minimum $f(0,0) = 4$, saddle points $\left(\pm\sqrt{2},-1\right)$
6. Saddle point $(1,2)$
7. Maximum $f(4,4) = 12$
8. Maximum $f\left(-\frac{1}{2},4\right) = -6$
9. Minima $f(-(1+y),y) = 0$, maximum $f(1,1) = 3$
10. Maximum $f(4,5) = 13$, minimum $f(4,0) = -7$
11. Maximum $f(-1,-2) = 17$, minimum $f(0,0) = 0$
12. Maximum $f\left(\frac{25}{4},5\right) = f\left(9,\frac{9}{2}\right) = \frac{45}{4}$, minimum $f(9,0) = -9$
13. Maximum $f(2,4) = 3$, minimum $f(-2,4) = -9$
14. Maximum $f(2,0) = 8$, minimum $f\left(-\frac{1}{4},0\right) = -\frac{17}{8}$
15. $\frac{7}{\sqrt{61}}$
16. $\left(-\frac{5}{3},-\frac{1}{6},\frac{25}{6}\right)$
17. $\left(\frac{2}{7},\frac{4}{7},\frac{6}{7}\right)$
18. $\dfrac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$

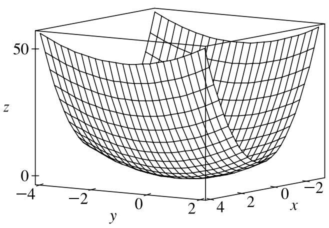
11.7 SOLUTIONS

E Click here for exercises.

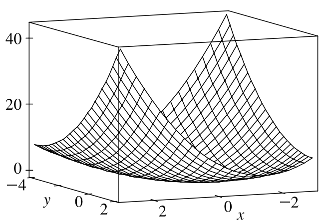
1. $f(x,y) = x^2 + y^2 + 4x - 6y \quad \Rightarrow \quad f_x = 2x + 4$, $f_y = 2y - 6$, $f_{xx} = f_{yy} = 2$, $f_{xy} = 0$. Then $f_x = 0$ and $f_y = 0$ implies $(x,y) = (-2,3)$ and $D(-2,3) = 4 > 0$, so $f(-2,3) = -13$ is a local minimum.



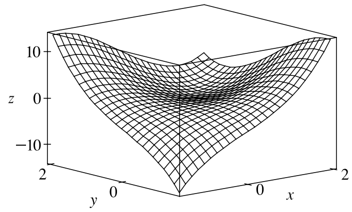
2. $f(x,y) = 4x^2 + y^2 - 4x + 2y \quad \Rightarrow \quad f_x = 8x - 4$, $f_y = 2y + 2$, $f_{xx} = 8$, $f_{yy} = 2$, $f_{xy} = 0$. Then $f_x = 0$ and $f_y = 0$ implies (x,y) is $\left(\frac{1}{2},-1\right)$ and $D\left(\frac{1}{2},-1\right) = 16 > 0$, so $f\left(\frac{1}{2},-1\right) = -2$ is a local minimum.



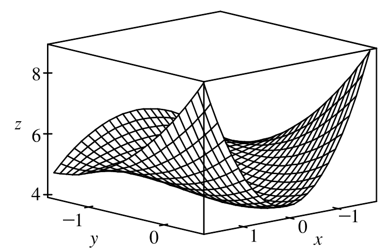
3. $f(x,y) = 2x^2 + y^2 + 2xy + 2x + 2y \quad \Rightarrow$ $f_x = 4x + 2y + 2$, $f_y = 2y + 2x + 2$, $f_{xx} = 4$, $f_{yy} = 2$, $f_{xy} = 2$. Then $f_x = 0$ and $f_y = 0$ implies $2x = 0$, so the critical point is $(0,-1)$. $D(0,-1) = 8 - 4 > 0$, so $f(0,-1) = -1$ is a local minimum.



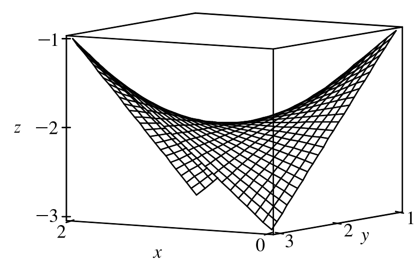
4. $f(x,y) = x^3 - 3xy + y^3 \quad \Rightarrow \quad f_x = 3x^2 - 3y$, $f_y = 3y^2 - 3x$, $f_{xx} = 6x$, $f_{yy} = 6y$, $f_{xy} = -3$. Then $f_x = 0$ implies $x^2 = y$ and substituting into $f_y = 0$ gives $x = 0$ or $x = 1$. Thus the critical points are $(0,0)$ and $(1,1)$. Now $D(0,0) = -9 < 0$ so $(0,0)$ is a saddle point and $D(1,1) = 36 - 9 > 0$ while $f_{xx}(1,1) = 6$ so $f(1,1) = -1$ is a local minimum.



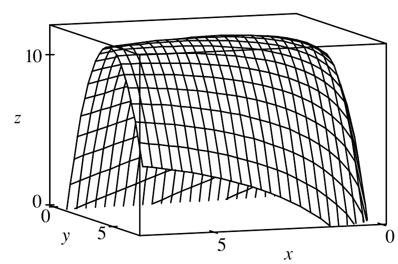
5. $f(x,y) = x^2 + y^2 + x^2y + 4 \quad \Rightarrow \quad f_x = 2x + 2xy$, $f_y = 2y + x^2$, $f_{xx} = 2 + 2y$, $f_{yy} = 2$, $f_{xy} = 2x$. Then $f_y = 0$ implies $y = -\frac{1}{2}x^2$, substituting into $f_x = 0$ gives $2x - x^3 = 0$ so $x = 0$ or $x = \pm\sqrt{2}$. Thus the critical points are $(0,0)$, $\left(\sqrt{2},-1\right)$ and $\left(-\sqrt{2},-1\right)$. Now $D(0,0) = 4$, $D\left(\sqrt{2},-1\right) = -8 = D\left(-\sqrt{2},-1\right)$, $f_{xx}(0,0) = 2$, $f_{xx}\left(\pm\sqrt{2},-1\right) = 0$. Thus $f(0,0) = 4$ is a local minimum and $\left(\pm\sqrt{2},-1\right)$ are saddle points.



6. $f(x,y) = xy - 2x - y \quad \Rightarrow \quad f_x = y - 2$, $f_y = x - 1$, $f_{xx} = f_{yy} = 0$, $f_{xy} = 1$ and the only critical point is $(1,2)$. Now $D(1,2) = -1$, so $(1,2)$ is a saddle point and f has no local maximum or minimum.



7. $f(x,y) = y\sqrt{x} - y^2 - x + 6y \quad \Rightarrow \quad f_x = y/\left(2\sqrt{x}\right) - 1$, $f_y = \sqrt{x} - 2y + 6$, $f_{yy} = -2$, $f_{xx} = -\frac{1}{4}yx^{-3/2}$, $f_{xy} = 1/\left(2\sqrt{x}\right)$. Then $f_x = 0$ implies $y = 2\sqrt{x}$ and substituting into $f_y = 0$ gives $-3\sqrt{x} + 6 = 0$ or $x = 4$. Thus the only critical point is $(4,4)$. $D(4,4) = -\frac{1}{8}(-2) - \left(\frac{1}{4}\right)^2 > 0$ and $f_{xx}(4,4) = -\frac{1}{8}$, so $f(4,4) = 12$ is a local maximum.



8. $f(x,y) = \dfrac{x^2y^2 - 8x + y}{xy} \quad \Rightarrow \quad f_x = y - x^{-2}$,

$f_y = x + 8y^{-2}$, $f_{xx} = 2x^{-3}$, $f_{yy} = -16y^{-3}$ and $f_{xy} = 1$.

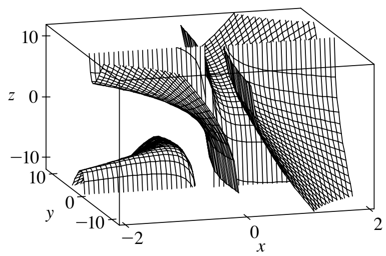
Then $f_x = 0$ implies $y = x^{-2}$, substituting into $f_y = 0$ gives

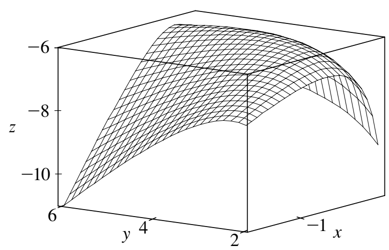
$x + 8x^4 = 0$ so $x = 0$ or $x = -\frac{1}{2}$ but $(0, y)$ is not in the

domain of f. Thus the only critical point is $\left(-\frac{1}{2}, 4\right)$. Then

$f_{xx}\left(-\frac{1}{2}, 4\right) = -16$ and $D\left(-\frac{1}{2}, 4\right) = 4 - 1 > 0$ so

$f\left(-\frac{1}{2}, 4\right) = -6$ is a local maximum.





9. $f(x,y) = \dfrac{(x+y+1)^2}{x^2+y^2+1} \quad \Rightarrow$

$$f_x = \frac{2(x+y+1)\left(x^2+y^2+1\right) - (x+y+1)^2(2x)}{\left(x^2+y^2+1\right)^2}$$

and $f_x = 0$ implies

$2(x+y+1)\left[\left(x^2+y^2+1\right) - (x+y+1)x\right] = 0$

or $(x+y+1)\left(y^2+1-xy-x\right) = 0$, so

$x = -(1+y)$ or $x = \dfrac{y^2+1}{y+1}$ (Note: In the latter $y \neq -1$; otherwise we get $0 = 2$.) Similarly

$$f_y = \frac{2(x+y+1)\left(x^2+y^2+1\right) - (x+y+1)^2(2y)}{\left(x^2+y^2+1\right)^2}$$

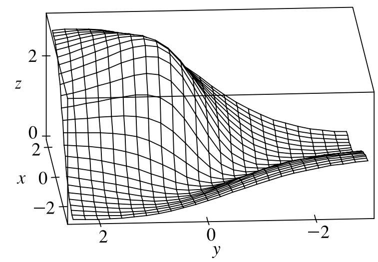
and $f_y = 0$ implies $(x+y+1)\left(x^2+1-xy-y\right) = 0$. Thus $x = -(1+y)$ also satisfies $f_y = 0$ and all points of the form $(-(1+y), y)$ are critical points. Substituting $x = \dfrac{y^2+1}{y+1}$ into $x^2+1-xy-y = 0$ and simplifying gives $-2y^3 + 2 = 0$ or $y = 1$ and $x = 1$. Note that $x = \dfrac{y^2+1}{y+1}$ is not a zero of $x+y+1$. So $(1,1)$ is the only other critical point. Now for each y, $f(-(1+y), y) = 0$ but $f(x,y) \geq 0$. Thus the points $(-(1+y), y)$ are local minima with value 0.

Also

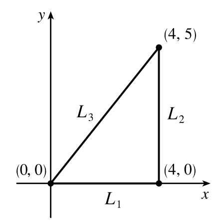
$$\begin{aligned}(x+y+1)^2 &= (x+y)^2 + 2(x+y) + 1 \\ &\leq 2x^2 + 2y^2 + 1 + 2(x+y) \\ &\leq 2x^2 + 2y^2 + 1 + \left(x^2+y^2+2\right) \\ &= 3\left(x^2+y^2+1\right)\end{aligned}$$

The last inequality is true because $0 \leq (x-1)^2 + (y-1)^2$.

Thus, $f(x,y) = \dfrac{(x+y+1)^2}{x^2+y^2+1} \leq 3$. But $f(1,1) = 3$ so $f(1,1) = 3$ is a local maximum.

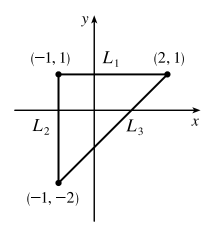


10. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist.

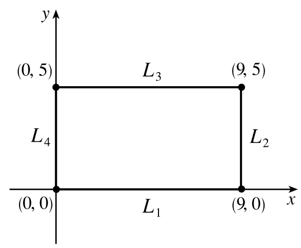


Here $f_x = -3$, $f_y = 4$ so there are no critical points inside D. Thus the absolute extrema must both occur on the boundary. Along L_1, $y = 0$ and $f(x, 0) = 5 - 3x$, a decreasing function in x, so the maximum value is $f(0,0) = 5$ and the minimum value is $f(4,0) = -7$. Along L_2, $x = 4$ and $f(4,y) = -7 + 4y$, an increasing function in y, so the minimum value is $f(4,0) = -7$ and the maximum value is $f(4,5) = 13$. Along L_3, $y = \frac{5}{4}x$ and $f\left(x, \frac{5}{4}x\right) = 5 + 2x$, an increasing function in x, so the minimum value is $f(0,0) = 5$ and the maximum value is $f(4,5) = 13$. Thus the absolute minimum of f on D is $f(4,0) = -7$ and the absolute maximum is $f(4,5) = 13$.

11. $f_x = 2x + 2y$ and $f_y = 2x + 6y$. Setting $f_x = f_y = 0$ gives $x = y = 0$ which yields the critical point $(0,0)$ where $f(0,0) = 0$. Along L_1: $y = 1$ and $f(x, 1) = x^2 + 2x + 3$, $-1 \leq x \leq 2$, which has a maximum at $x = 2$ where $f(2,1) = 11$, and a minimum at $x = -1$ where $f(-1,1) = 2$. Along L_2: $x = -1$ and $f(-1,y) = 1 - 2y + 3y^2$, $-2 \leq y \leq 1$, which has a maximum at $y = -2$ where $f(-1,-2) = 17$ and a minimum at $y = \frac{1}{3}$ where $f\left(-1, \frac{1}{3}\right) = \frac{2}{3}$. Along L_3: $y = x - 1$ and $f(x, x-1) = 6x^2 - 8x + 3$, $-1 \leq x \leq 2$, which has a maximum at $x = -1$ where $f(-1,-2) = 17$ and a minimum at $x = \frac{2}{3}$ where $f\left(\frac{2}{3}, -\frac{1}{3}\right) = \frac{1}{3}$. As a result, the absolute maximum value of f on D is $f(-1,-2) = 17$ and the minimum value is $f(0,0) = 0$.

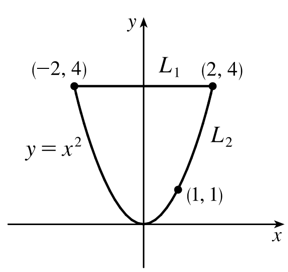


12. Since $x \geq 0$ in D, f is continuous on D. In Problem 7 we found that the only critical point of f is $(4, 4)$ and $f(4, 4) = -\frac{1}{8}$. [Note that $(4, 4)$ is in D.]



On L_1: $f(x, 0) = -x$, so the maximum value is $f(0, 0) = 0$ and the minimum value is $f(9, 0) = -9$. On L_2: $f(9, y) = 9y - y^2 - 9$, a quadratic in y which attains its maximum at $y = \frac{9}{2}$, $f\left(9, \frac{9}{2}\right) = \frac{45}{4}$ and its minimum at $y = 0$, $f(9, 0) = -9$. On L_3: $f(x, 5) = 5\sqrt{x} - x + 5$, a function whose maximum is attained at $x = \frac{25}{4}$, $f\left(\frac{25}{4}, 5\right) = \frac{45}{4}$ and its minimum at $x = 0$, $f(0, 5) = 5$. On L_4: $f(0, y) = -y^2 + 6y$, a quadratic in y which attains its maximum at $y = 3$, $f(0, 3) = 9$ and its minimum at $y = 0$, $f(0, 0) = 0$. Thus the absolute maximum of f on D is $f\left(\frac{25}{4}, 5\right) = f\left(9, \frac{9}{2}\right) = \frac{45}{4}$ and the absolute minimum is $f(9, 0) = -9$.

13. $f_x(x, y) = y - 1$ and $f_y(x, y) = x - 1$ and so the critical point is $(1, 1)$ (in D), where $f(1, 1) = 0$.



Along L_1: $y = 4$, so $f(x, 4) = 1 + 4x - x - 4 = 3x - 3$, $-2 \leq x \leq 2$, which is an increasing function and has a maximum value when $x = 2$ where $f(2, 4) = 3$ and a minimum of $f(-2, 4) = -9$. Along L_2: $y = x^2$, so let $g(x) = f(x, x^2) = x^3 - x^2 - x + 1$. Then $g'(x) = 3x^2 - 2x - 1 = 0 \quad \Leftrightarrow \quad x = -\frac{1}{3}$ or $x = 1$. $f\left(-\frac{1}{3}, \frac{1}{9}\right) = \frac{32}{27}$ and $f(1, 1) = 0$. As a result, the absolute maximum and minimum values of f on D are $f(2, 4) = 3$ and $f(-2, 4) = -9$.

14. $f_x = 4x + 1$, $f_y = 2y$ and the only critical point is $\left(-\frac{1}{4}, 0\right)$ (and this point is in D) and $f\left(-\frac{1}{4}, 0\right) = -\frac{17}{8}$. On the circle $x^2 + y^2 = 4$, $f(x, y) = x^2 + x + 2$, a quadratic in x which attains its minimum at $\left(-\frac{1}{2}, \pm\frac{\sqrt{15}}{2}\right)$, $f\left(-\frac{1}{2}, \pm\frac{\sqrt{15}}{2}\right) = \frac{7}{4}$ and its maximum at $(2, 0)$, $f(2, 0) = 8$. Thus the absolute maximum of f on D is $f(2, 0) = 8$ while the absolute minimum is $f\left(-\frac{1}{4}, 0\right) = -\frac{17}{8}$.

15. $d = \sqrt{(x-2)^2 + (y+2)^2 + (z-3)^2}$, where $z = \frac{1}{3}(6x + 4y - 2)$, so we minimize $d^2 = f(x, y) = (x-2)^2 + (y+2)^2 + \left(2x + \frac{4}{3}y - \frac{11}{3}\right)^2$. Then $f_x = 10x + \frac{16}{3}y - \frac{56}{3}$ and $f_y = \frac{50}{9}y + \frac{16}{3}x - \frac{52}{9}$. Solving $50y + 48x = 52$ and $16y + 30x = 56$ simultaneously gives $x = \frac{164}{61}$, $y = -\frac{94}{61}$. The absolute minimum must occur at a critical point. Thus $d^2 = \left(\frac{42}{61}\right)^2 + \left(\frac{28}{61}\right)^2 + \left(-\frac{21}{61}\right)^2$ or $d = \frac{7}{\sqrt{61}}$.

16. Here $d = \sqrt{(x+4)^2 + (y-1)^2 + (z-3)^2}$, where $z = 1 + y - 2x$. So we minimize $d^2 = f(x, y) = (x+4)^2 + (y-1)^2 + (-2 - 2x + y)^2$. Then $f_x = 2(x+4) - 4(-2 - 2x + y) = 10x - 4y + 16 = 0$ implies $y = \frac{5}{2}x + 4$ and $f_y = 4y - 4x - 6 = 0$, so the only critical point is $\left(-\frac{5}{3}, -\frac{1}{6}\right)$. Thus the closest point to $(-4, 1, 3)$ is $\left(-\frac{5}{3}, -\frac{1}{6}, \frac{25}{6}\right)$.

17. Here $d = \sqrt{x^2 + y^2 + z^2}$ with $z = \frac{1}{3}(4 - x - 2y)$. So minimize $d^2 = x^2 + y^2 + \frac{1}{9}(4 - x - 2y)^2 = f(x, y)$. Then $f_x = 2x - \frac{2}{9}(4 - x - 2y) = \frac{1}{9}(20x + 4y - 8)$, $f_y = \frac{1}{9}(26y + 4x - 16)$. Solving the equations $20x + 4y - 8 = 0$ and $26y + 4x - 16 = 0$, we get $x = \frac{2}{7}$, $y = \frac{4}{7}$, so the only critical point is $\left(\frac{2}{7}, \frac{4}{7}\right)$. Since the absolute minimum has to occur at a critical point, the point $\left(\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right)$ is is the closest point to the origin, or check $D\left(\frac{2}{7}, \frac{4}{7}\right)$.

18. Here $z = -\dfrac{Ax + By + D}{C}$ and since A, B, C, D, x_0, y_0, and z_0 are constants, there exists a real number $\alpha = -Ax_0 - By_0 - Cz_0 - D$ and $z - z_0 = -\dfrac{A(x - x_0) + B(y - y_0) - \alpha}{C}$.
Then setting $X = x - x_0$ and $Y = y - y_0$ (so $dX/dx = dY/dy = 1$), we need to minimize
$f(X, Y) = X^2 + Y^2 + \dfrac{1}{C^2}(AX + BY - \alpha)^2$.
Here $f_x = 2X + \dfrac{2A}{C^2}(AX + BY - \alpha) = 0$ implies $X = A\dfrac{\alpha - BY}{A^2 + C^2}$ and
$f_y = 2Y + \dfrac{2B}{C^2}(AX + BY - \alpha) = 0$
implies (with the X above) that
$(C^2 + B^2)Y + \dfrac{BA^2}{A^2 + C^2}(\alpha - BY) - B\alpha = 0$ or
$Y = \dfrac{A^2 + C^2}{C^2(A^2 + B^2 + C^2)}\dfrac{BC^2}{A^2 + C^2}\alpha = \dfrac{B}{A^2 + B^2 + C^2}\alpha.$

Thus

$$X = \frac{A}{A^2 + C^2}\left(1 - \frac{B^2}{A^2 + B^2 + C^2}\right)\alpha = \frac{A}{A^2 + B^2 + C^2}\alpha$$

and

$$z - z_0 = \frac{1}{C}\left[\frac{A^2}{A^2 + B^2 + C^2} + \frac{B^2}{A^2 + B^2 + C^2} - 1\right]\alpha = -\frac{C}{A^2 + B^2 + C^2}\alpha$$

Finally the minimum distance is

$$d = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = \frac{\sqrt{(A^2 + B^2 + C^2)\alpha^2}}{A^2 + B^2 + C^2} = \frac{|\alpha|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Compare with the result of Example 10.5.7.