12.3 DOUBLE INTEGRALS IN POLAR COORDINATES

A Click here for answers.

S Click here for solutions.

1–9 ▪ Evaluate the given integral by changing to polar coordinates.

1. $\iint_R x\,dA$, where R is the disk with center the origin and radius 5

2. $\iint_R y\,dA$, where R is the region in the first quadrant bounded by the circle $x^2 + y^2 = 9$ and the lines $y = x$ and $y = 0$

3. $\iint_R xy\,dA$, where R is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 25$

4. $\iint_R \sqrt{x^2 + y^2}\,dA$, where $R = \{(x, y) \mid 1 \leqslant x^2 + y^2 \leqslant 9,\ y \geqslant 0\}$

5. $\iint_R \sin(x^2 + y^2)\,dA$, where R is the annular region $1 \leqslant x^2 + y^2 \leqslant 16$

6. $\iint_D 1/\sqrt{x^2 + y^2}\,dA$, where D is the region that lies inside the cardioid $r = 1 + \sin\theta$ and outside the circle $r = 1$

7. $\iint_D \sqrt{x^2 + y^2}\,dA$, where D is the region bounded by the cardioid $r = 1 + \cos\theta$

8. $\displaystyle\iint_D \frac{1}{(1 + x^2 + y^2)^{3/2}}\,dA$, where D is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 16$

9. $\iint_D (x^2 + y^2)\,dA$, where D is the region bounded by the spirals $r = \theta$ and $r = 2\theta$ for $0 \leqslant \theta \leqslant 2\pi$

10–13 ▪ Use a double integral to find the area of the region.

10. The region enclosed by the cardioid $r = 1 - \sin\theta$

11. The region enclosed by the lemniscate $r^2 = 4\cos 2\theta$

12. The region inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$

13. The smaller region bounded by the spiral $r\theta = 1$, the circles $r = 1$ and $r = 3$, and the polar axis

14–17 ▪ Use polar coordinates to find the volume of the given solid.

14. Under the cone $z = \sqrt{x^2 + y^2}$ and above the ring $4 \leqslant x^2 + y^2 \leqslant 25$

15. Under the plane $6x + 4y + z = 12$ and above the disk with boundary circle $x^2 + y^2 = y$

16. Inside the sphere $x^2 + y^2 + z^2 = 4a^2$ and outside the cylinder $x^2 + y^2 = 2ax$

17. A sphere of radius a

18. Evaluate the iterated integral

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \arctan\frac{y}{x}\,dy\,dx$$

by converting to polar coordinates.

12.3 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. 0
2. $9\left(1-\frac{1}{\sqrt{2}}\right)$
3. $\frac{609}{8}$
4. $\frac{26}{3}\pi$
5. $\pi\left(\cos 1-\cos 16\right)$
6. 2
7. $\frac{5\pi}{3}$
8. $\frac{\pi}{2}\left(1-\frac{1}{\sqrt{17}}\right)$
9. $24\pi^5$
10. $\frac{3\pi}{2}$
11. 4
12. π
13. 2
14. 78π
15. $\frac{5\pi}{2}$
16. $16a^3\left(\frac{3\pi+4}{9}\right)$
17. $\frac{4\pi}{3}a^3$
18. $\frac{9}{4}\pi^2$

12.3 SOLUTIONS

E Click here for exercises.

1. The region R can be described in polar coordinates as $R = \{(r, \theta) \mid 0 \le \theta \le 2\pi, 0 \le r \le 5\}$. Thus,

$$\iint_R x\,dA = \int_0^{2\pi}\int_0^5 (r\cos\theta)\,r\,dr\,d\theta = \left(\int_0^{2\pi}\cos\theta\,d\theta\right)\left(\int_0^5 r^2\,dr\right) = [\sin\theta]_0^{2\pi}\left[\tfrac{1}{3}r^3\right]_0^5 = 0$$

2. $$\iint_R y\,dA = \int_0^{\pi/4}\int_0^3 (r\sin\theta)\,r\,dr\,d\theta = \left(\int_0^{\pi/4}\sin\theta\,d\theta\right)\left(\int_0^3 r^2\,dr\right) = \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)(9) = 9\left(1 - \frac{1}{\sqrt{2}}\right)$$

3. $$\iint_R xy\,dA = \int_0^{\pi/2}\int_2^5 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta = \int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\int_2^5 r^3\,dr = \left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2}\left[\tfrac{1}{4}r^4\right]_2^5 = \tfrac{1}{2}\cdot\tfrac{5^4-2^4}{4} = \tfrac{609}{8}$$

4. $$\iint_R \sqrt{x^2+y^2}\,dA = \int_0^{\pi}\int_1^3 \sqrt{r^2}\,r\,dr\,d\theta = \left(\int_0^{\pi}d\theta\right)\left(\int_1^3 r^2\,dr\right) = [\theta]_0^{\pi}\left[\tfrac{1}{3}r^3\right]_1^3 = \pi\left(\tfrac{27-1}{3}\right) = \tfrac{26}{3}\pi$$

5. $$\int_0^{2\pi}\int_1^4 r\sin\left(r^2\right)dr\,d\theta = -2\pi\left[\tfrac{1}{2}\cos\left(r^2\right)\right]_1^4 = \pi(\cos 1 - \cos 16)$$

6. The circle $r = 1$ intersects the cardioid $r = 1 + \sin\theta$ when $1 = 1 + \sin\theta \;\Rightarrow\; \theta = 0$ or $\theta = \pi$, so

$$\iint_D \frac{1}{\sqrt{x^2+y^2}}\,dA = \int_0^{\pi}\int_1^{1+\sin\theta}\left(\frac{1}{r}\right)r\,dr\,d\theta = \int_0^{\pi}[r]_1^{1+\sin\theta}\,d\theta = \int_0^{\pi}\sin\theta\,d\theta = [-\cos\theta]_0^{\pi} = 2$$

7. $$\int_0^{2\pi}\int_0^{1+\cos\theta} r^2\,dr\,d\theta = \int_0^{2\pi}\tfrac{1}{3}(1+\cos\theta)^3\,d\theta = \tfrac{1}{3}\int_0^{2\pi}\left(\cos^3\theta + 3\cos^2\theta + 3\cos\theta + 1\right)d\theta = \tfrac{1}{3}(0 + 3\pi + 0 + 2\pi) = \tfrac{5\pi}{3}$$

8. $$\iint_D \frac{1}{(1+x^2+y^2)^{3/2}}\,dA = \int_0^{\pi/2}\int_0^4 \frac{1}{(1+r^2)^{3/2}}\,r\,dr\,d\theta = \left(\int_0^{\pi/2}d\theta\right)\left(\int_0^4 r\left(1+r^2\right)^{-3/2}dr\right) = [\theta]_0^{\pi/2}\left[-\left(1+r^2\right)^{-1/2}\right]_0^4 = \tfrac{\pi}{2}\left(1 - \tfrac{1}{\sqrt{17}}\right)$$

9. $$\int_0^{2\pi}\int_\theta^{2\theta} r^2\,r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_{r=\theta}^{r=2\theta}d\theta = \tfrac{1}{4}\int_0^{2\pi}15\theta^4\,d\theta = \tfrac{3}{4}\left[\theta^5\right]_0^{2\pi} = \tfrac{3}{4}\left(32\pi^5\right) = 24\pi^5$$

10. By symmetry,

$$A = 2\int_{-\pi/2}^{\pi/2}\int_0^{1-\sin\theta} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[r^2\right]_{r=0}^{r=1-\sin\theta}d\theta = \int_{-\pi/2}^{\pi/2}\left(1 - 2\sin\theta + \sin^2\theta\right)d\theta = \int_{-\pi/2}^{\pi/2}\left[1 + \tfrac{1}{2}(1-\cos 2\theta)\right]d\theta = \int_{-\pi/2}^{\pi/2}\left(\tfrac{3}{2} - \tfrac{1}{2}\cos 2\theta\right)d\theta$$

since $2\sin\theta$ is an odd function. But $\frac{3}{2} - \frac{1}{2}\cos 2\theta$ is an even function, so $A = \int_0^{\pi/2}(3 - \cos 2\theta)\,d\theta = \left[3\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{3\pi}{2}$.

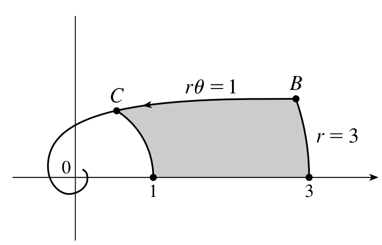
11. By symmetry, the two loops of the lemniscate are equal in area, so

$$A = 2\int_{-\pi/4}^{\pi/4}\int_0^{2\sqrt{\cos 2\theta}} r\,dr\,d\theta = \int_{-\pi/4}^{\pi/4}\left[r^2\right]_{r=0}^{r=2\sqrt{\cos 2\theta}}d\theta = \int_{-\pi/4}^{\pi/4}4\cos 2\theta\,d\theta = 8\int_0^{\pi/4}\cos 2\theta\,d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4$$

12. $3\cos\theta = 1 + \cos\theta$ implies $\cos\theta = \frac{1}{2}$, so $\theta = \pm\frac{\pi}{3}$. Then by symmetry

$$A = 2\int_0^{\pi/3}\int_{1+\cos\theta}^{3\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/3}\left[\tfrac{1}{2}r^2\right]_{1+\cos\theta}^{3\cos\theta}d\theta = \int_0^{\pi/3}\left(9\cos^2\theta - 1 - 2\cos\theta - \cos^2\theta\right)d\theta = \int_0^{\pi/3}\left[8\cdot\tfrac{1}{2}(1+\cos 2\theta) - 2\cos\theta - 1\right]d\theta = [4\theta + 2\sin 2\theta - 2\sin\theta - \theta]_0^{\pi/3} = \pi$$

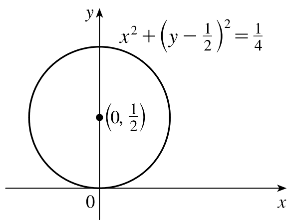
13.



C is the point $r = 1$, $\theta = 1$ rad; the arrow indicates the direction of increasing θ. B is the point $r = 3$, $\theta = \frac{1}{3}$ rad. The region, as a type I polar region, is $\{(r, \theta) \mid 1 \le r \le 3, 0 \le \theta \le 1/r\}$, so by Formula 3, we have $A = \int_1^3\int_0^{1/r} r\,d\theta\,dr = \int_1^3 1\,dr = 2$.

14. $$V = \iint_{4\le x^2+y^2\le 25}\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_2^5 r^2\,dr\,d\theta = 2\pi\cdot\tfrac{5^3-2^3}{3} = \tfrac{234\pi}{3} = 78\pi$$

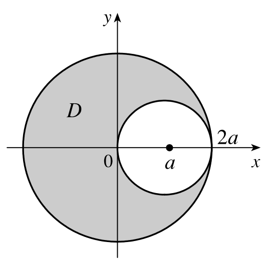
15.



$$V = \iint_{x^2+\left(y-\frac{1}{2}\right)^2\le\frac{1}{4}} (12-6x-4y)\,dA$$
$$= \int_0^\pi \int_0^{\sin\theta} \left(12r-6r^2\cos\theta-4r^2\sin\theta\right) dr\,d\theta$$
$$= \int_0^\pi \left(6\sin^2\theta-2\sin^3\theta\cos\theta-\tfrac{4}{3}\sin^4\theta\right) d\theta$$
$$= \left[\tfrac{5}{2}(\theta-\sin\theta\cos\theta)-\tfrac{1}{2}\sin^4\theta+\tfrac{1}{3}\sin^2\theta\cos\theta\right]_0^\pi = \tfrac{5\pi}{2}$$

This can also be done without calculus: the volume of this cylinder is $\pi\left(\frac{1}{4}\right)\left(\frac{12+8}{2}\right)=\frac{5\pi}{2}$.

16.



$x^2+y^2=2ax$ implies $(x-a)^2+y^2=a^2$ is the equation of the cylinder. Thus the region in the xy-plane is the disk $x^2+y^2\le 4a^2$ less the disk $(x-a)^2+y^2\le a^2$. In polar coordinates $x^2+y^2=2ax$ becomes $r=2a\cos\theta$ and the desired volume is the volume of the sphere less the volume of the cylinder inside the sphere. Moreover, by symmetry the volume of the cylinder inside the sphere is twice that above the xy-plane. Hence

$$V = \tfrac{32\pi}{3}a^3 - 2\iint_{(x-a)^2+y^2\le a^2}\sqrt{4a^2-x^2-y^2}\,dA$$
$$= \tfrac{32\pi}{3}a^3 - 2\int_{-\pi/2}^{\pi/2}\int_0^{2a\cos\theta}\sqrt{4a^2-r^2}\,r\,dr\,d\theta$$
$$= \tfrac{32\pi}{3}a^3 - 2\int_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{3}\left(4a^2-r^2\right)^{3/2}\right]_0^{2a\cos\theta} d\theta$$
$$= \tfrac{32\pi}{3}a^3 - \tfrac{2}{3}\int_{-\pi/2}^{\pi/2} 8a^3\left[1-\left(1-\cos^2\theta\right)^{3/2}\right] d\theta$$
$$= \tfrac{32\pi}{3}a^3 - \tfrac{16}{3}a^3\int_{-\pi/2}^{\pi/2}\left(1-|\sin\theta|^3\right) d\theta$$

But

$$\int_{-\pi/2}^{\pi/2}|\sin\theta|^3 d\theta = \int_0^{\pi/2}\sin^3\theta\,d\theta + \int_{-\pi/2}^{0} -\sin^3\theta\,d\theta$$
$$= \left[-\cos\theta+\tfrac{1}{3}\cos^3\theta\right]_0^{\pi/2} - \left[-\cos\theta+\tfrac{1}{3}\cos^3\theta\right]_{-\pi/2}^{0}$$
$$= \tfrac{4}{3}$$

Thus

$$V = \tfrac{32\pi}{3}a^3 - \tfrac{16}{3}a^3\left(\pi-\tfrac{4}{3}\right) = \tfrac{16\pi}{3}a^3 + \tfrac{64}{9}a^3$$
$$= 16a^3\left(\tfrac{3\pi+4}{9}\right)$$

17. $V = 2\iint_{x^2+y^2\,\le\,a^2}\sqrt{a^2-x^2-y^2}\,dA$

$$= 2\int_0^{2\pi}\int_0^a \sqrt{a^2-r^2}\,r\,dr\,d\theta$$
$$= \tfrac{4\pi}{3}\left[-\left(a^2-r^2\right)^{3/2}\right]_0^a = \tfrac{4\pi}{3}a^3$$

18. $\int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \arctan(y/x)\,dy\,dx$

$$= \int_0^\pi\int_0^3 \arctan(\tan\theta)\,r\,dr\,d\theta = \int_0^\pi\int_0^3 \theta r\,dr\,d\theta$$
$$= \int_0^\pi \left[\tfrac{1}{2}r^2\right]_0^3 d\theta = \tfrac{9}{4}\theta^2\Big]_0^\pi = \tfrac{9}{4}\pi^2$$