13.2 LINE INTEGRALS

A Click here for answers.

S Click here for solutions.

1–14 ▪ Evaluate the line integral, where C is the given curve.

1. $\int_C x\,ds,\quad C\colon x = t^3,\ y = t,\ 0 \leqslant t \leqslant 1$

2. $\int_C y\,ds,\quad C\colon x = t^3,\ y = t^2,\ 0 \leqslant t \leqslant 1$

3. $\int_C xy\,ds,\quad C$ is the line segment joining $(-1, 1)$ to $(2, 3)$

4. $\int_C (x - 2y^2)\,dy,$
C is the arc of the parabola $y = x^2$ from $(-2, 4)$ to $(1, 1)$

5. $\int_C \sin x\,dx,$
C is the arc of the curve $x = y^4$ from $(1, -1)$ to $(1, 1)$

6. $\int_C x\sqrt{y}\,dx + 2y\sqrt{x}\,dy,$
C consists of the shortest arc of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$ and the line segment from $(0, 1)$ to $(4, 3)$

7. $\int_C xyz\,ds,$
$C\colon x = 2t,\ y = 3\sin t,\ z = 3\cos t,\ 0 \leqslant t \leqslant \pi/2$

8. $\int_C x^2 z\,ds,$
$C\colon x = \sin 2t,\ y = 3t,\ z = \cos 2t,\ 0 \leqslant t \leqslant \pi/4$

9. $\int_C xy^2 z\,ds,$
C is the line segment from $(1, 0, 1)$ to $(0, 3, 6)$

10. $\int_C xz\,ds,\quad C\colon x = 6t,\ y = 3\sqrt{2}\,t^2,\ z = 2t^3,\ 0 \leqslant t \leqslant 1$

11. $\int_C x^3 y^2 z\,dz,\quad C\colon x = 2t,\ y = t^2,\ z = t^2,\ 0 \leqslant t \leqslant 1$

12. $\int_C yz\,dy + xy\,dz,\quad C\colon x = \sqrt{t},\ y = t,\ z = t^2,\ 0 \leqslant t \leqslant 1$

13. $\int_C z^2\,dx - z\,dy + 2y\,dz,$
C consists of line segments from $(0, 0, 0)$ to $(0, 1, 1)$, from $(0, 1, 1)$ to $(1, 2, 3)$, and from $(1, 2, 3)$ to $(1, 2, 4)$

14. $\int_C yz\,dx + xz\,dy + xy\,dz,$
C consists of line segments from $(0, 0, 0)$ to $(2, 0, 0)$, from $(2, 0, 0)$ to $(1, 3, -1)$, and from $(1, 3, -1)$ to $(1, 3, 0)$

15–17 ▪ Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the vector function $\mathbf{r}(t)$.

15. $\mathbf{F}(x, y) = x^2 y\,\mathbf{i} - xy\,\mathbf{j},$
$\mathbf{r}(t) = t^3\,\mathbf{i} + t^4\,\mathbf{j},\quad 0 \leqslant t \leqslant 1$

16. $\mathbf{F}(x, y, z) = (y + z)\,\mathbf{i} - x^2\,\mathbf{j} - 4y^2\,\mathbf{k},$
$\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^4\,\mathbf{k},\quad 0 \leqslant t \leqslant 1$

17. $\mathbf{F}(x, y, z) = x^2\,\mathbf{i} + xy\,\mathbf{j} + z^2\,\mathbf{k},$
$\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t^2\,\mathbf{k},\quad 0 \leqslant t \leqslant \pi/2$

18–19 ▪ Use a calculator to find the integral to three decimal places.

18. $\int x \sin y\,ds,\quad C\colon x = \ln t,\ y = e^{-t},\ 1 \leqslant t \leqslant 2$

19. $\int z^2 \ln(1 + x^2 + y^2)\,ds,\quad C\colon x = t,\ y = t^2,\ z = t^3,\ 0 \leqslant t \leqslant 1$

20. Find the work done by the force field

$$\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yx\,\mathbf{j} + zy\,\mathbf{k}$$

on a particle that moves along the curve

$$\mathbf{r}(t) = t^2\,\mathbf{i} - t^3\,\mathbf{j} + t^4\,\mathbf{k},\quad 0 \leqslant t \leqslant 1$$

13.2 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $\frac{1}{54}\left(10^{3/2}-1\right)$
2. $\frac{1}{1215}\left[19\,(13)^{3/2}+64\right]$
3. $\frac{3\sqrt{13}}{2}$
4. 48
5. 0
6. $\frac{32\sqrt{3}+66}{5}$
7. $\frac{9\sqrt{13}}{4}\pi$
8. $\frac{\sqrt{13}}{6}$
9. $3\sqrt{35}$
10. $\frac{864}{35}$
11. $\frac{16}{11}$
12. $\frac{23}{28}$
13. $\frac{77}{6}$
14. 0
15. $-\frac{19}{143}$
16. $-\frac{59}{30}$
17. $\dfrac{\pi^6}{192}$
18. 0.052
19. 0.396
20. $\frac{23}{88}$

13.2 SOLUTIONS

E Click here for exercises.

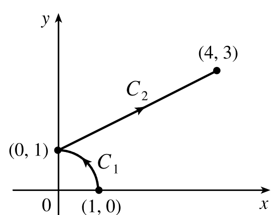
1. $\int_C x\,ds = \int_0^1 \left(t^3\right)\sqrt{9t^4+1}\,dt = \frac{1}{54}\left(9t^4+1\right)^{3/2}\Big|_0^1$
$= \frac{1}{54}\left(10^{3/2}-1\right)$

2. $\int_C y\,ds = \int_0^1 \left(t^2\right)\sqrt{9t^4+4t^2}\,dt = \int_0^1 t^3\sqrt{9t^2+4}\,dt$
$= \int_0^1 3t^3\sqrt{t^2+\left(\frac{2}{3}\right)^2}\,dt$
$= 3\left(\frac{1}{5}t^2-\frac{8}{135}\right)\left(t^2+\frac{4}{9}\right)^{3/2}\Big|_0^1$
$= \frac{19}{45}\left(\frac{13}{9}\right)^{3/2}+\frac{8}{45}\left(\frac{8}{27}\right)$
$= \frac{1}{1215}\left[19\left(13\right)^{3/2}+64\right]$

3. The line is $3y-2x=5$, so $x=x$, $y=\frac{1}{3}\left(2x+5\right)$, $-1\leq x\leq 2$. Then
$\int_C xy\,ds = \int_{-1}^2 \frac{1}{3}x\left(2x+5\right)\cdot\frac{\sqrt{13}}{3}\,dx$
$= \frac{\sqrt{13}}{9}\int_{-1}^2\left(2x^2+5x\right)dx = \frac{\sqrt{13}}{9}\left(\frac{46}{3}-\frac{11}{6}\right) = \frac{3\sqrt{13}}{2}$

4. $x=x$, $y=x^2$, $-2\leq x\leq 1$. Then
$\int_C \left(x-2y^2\right)dy = \int_{-2}^1\left(x-2x^4\right)2x\,dx$
$= \int_{-2}^1\left(2x^2-4x^5\right)dx = \frac{2}{3}\left[x^3-x^6\right]_{-2}^1 = 48$

5. Choosing y as the parameter, we have $x=y^4$, $y=y$, $-1\leq y\leq 1$. Then
$\int_C \sin x\,dx = \int_{-1}^1\left(\sin y^4\right)\left(4y^3\right)dy = -\cos y^4\Big]_{-1}^1 = 0.$

6. 

On C_1: $x=\cos t \;\Rightarrow\; dx=-\sin t\,dt$, $y=\sin t \;\Rightarrow\; y=\cos t\,dt$, $0\leq t\leq\frac{\pi}{2}$.
On C_2: $x=4t \;\Rightarrow\; dx=4\,dt$, $y=2t+1 \;\Rightarrow\; dy=2\,dt$, $0\leq t\leq 1$. Then
$\int_C x\sqrt{y}\,dx+2y\sqrt{x}\,dy$
$= \int_{C_1} x\sqrt{y}\,dx+2y\sqrt{x}\,dy+\int_{C_2} x\sqrt{y}\,dx+2y\sqrt{x}\,dy$
$= \int_0^{\pi/2}\left[-\cos t\left(\sin t\right)^{3/2}+2\sin t\left(\cos t\right)^{3/2}\right]dt$
$+\int_0^1\left[16t\sqrt{2t+1}+8\left(2t+1\right)\sqrt{t}\right]dt$
$= \left[-\frac{2}{5}\left(\sin t\right)^{5/2}-\frac{4}{5}\left(\cos t\right)^{5/2}\right]_0^{\pi/2}$
$+\left[\frac{16}{3}t\left(2t+1\right)^{3/2}-\frac{16}{15}\left(2t+1\right)^{5/2}\right.$
$\left.+8\left(\frac{4}{5}t^{5/2}+\frac{2}{3}t^{3/2}\right)\right]_0^1$
$= \frac{2}{5}+\frac{16}{3}\cdot 3\sqrt{3}-\frac{16}{15}\cdot 3^2\cdot\sqrt{3}+\frac{16}{15}+8\left(\frac{4}{5}+\frac{2}{3}\right)$
$= \frac{32\sqrt{3}+66}{5}$

7. $\int_C xyz\,ds = \int_0^{\pi/2}\left(18t\sin t\cos t\right)\sqrt{4+9}\,dt$
$= 18\sqrt{13}\int_0^{\pi/2}\left(t\sin t\cos t\right)dt$
$= 18\sqrt{13}\int_0^{\pi/2}\frac{1}{2}t\sin 2t\,dt$
$= 9\sqrt{13}\left[-\frac{1}{2}t\cos 2t+\frac{1}{4}\sin 2t\right]_0^{\pi/2} = \frac{9\sqrt{13}}{4}\pi$

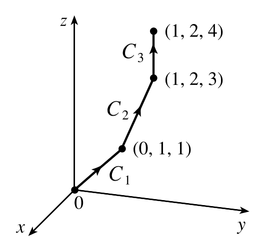
8. $\int_C x^2z\,ds = \int_0^{\pi/4}\left(\sin^2 2t\cos 2t\right)\sqrt{4+9}\,dt$
$= \sqrt{13}\left[\frac{1}{6}\sin^3 2t\right]_0^{\pi/4} = \frac{\sqrt{13}}{6}$

9. $x=-t+1$, $y=3t$, $z=5t+1$, $0\leq t\leq 1$.
$\int_C xy^2z\,ds = \int_0^1\left(1-t\right)\left(9t^2\right)\left(5t+1\right)\sqrt{1+9+25}\,dt$
$= 9\sqrt{35}\int_0^1\left(t^2+4t^3-5t^4\right)dt = 3\sqrt{35}$

10. $\sqrt{\left(dx/dt\right)^2+\left(dy/dt\right)^2+\left(dz/dt\right)^2}$
$= \sqrt{36+36\left(2\right)t^2+36t^4} = 6\sqrt{\left(t^2+1\right)^2} = 6\left(t^2+1\right)$
Then
$\int_C xz\,ds = \int_0^1 72t^4\left(t^2+1\right)dt = 72\left[\frac{1}{7}t^7+\frac{1}{5}t^5\right]_0^1$
$= 72\left(\frac{12}{35}\right) = \frac{864}{35}$

11. $\int_C x^3y^2z\,dz = \int_0^1\left(2t\right)^3\left(t^2\right)^2\left(t^2\right)\left(2t\right)dt$
$= \int_0^1\left(8t^3\right)\left(t^4\right)\left(t^2\right)\left(2t\right)dt = \int_0^1 16t^{10}\,dt$
$= \frac{16}{11}t^{11}\Big]_0^1 = \frac{16}{11}$

12. $\int_C yz\,dy+xy\,dz = \int_0^1\left(t\right)\left(t^2\right)dt+\int_0^1\sqrt{t}\left(t\right)2t\,dt$
$= \int_0^1\left(t^3+2t^{5/2}\right)dt = \left[\frac{1}{4}t^4+\frac{4}{7}t^{7/2}\right]_0^1 = \frac{23}{28}$

13. 

On C_1: $x=0 \;\Rightarrow\; dx=0\,dt$, $y=t \;\Rightarrow\; dy=dt$, $z=t \;\Rightarrow\; dz=dt$, $0\leq t\leq 1$.
On C_2: $x=t \;\Rightarrow\; dx=dt$, $y=t+1 \;\Rightarrow\; dy=dt$, $z=2t+1 \;\Rightarrow\; dz=2\,dt$, $0\leq t\leq 1$.
On C_3: $x=1 \;\Rightarrow\; dx=0\,dt$, $y=2 \;\Rightarrow\; dy=0\,dt$, $z=t+3 \;\Rightarrow\; dz=dt$, $0\leq t\leq 1$.
Then
$\int_C z^2\,dx-z\,dy+2y\,dz$
$= \int_0^1\left(0-t+2t\right)dt$
$+\int_0^1\left[\left(2t+1\right)^2-\left(2t+1\right)+2\left(t+1\right)\left(2\right)\right]dt$
$+\int_0^1\left(0+0+4\right)dt$
$= \frac{1}{2}+\left[\frac{4}{3}t^3+3t^2+4t\right]_0^1+4 = \frac{77}{6}$

14. C_1: $(0,0,0)$ to $(2,0,0)$: $x = 2t$, $y = z = 0$, $0 \le t \le 1$.
C_2: $(2,0,0)$ to $(1,3,-1)$: $x = -t+2$, $y = 3t$, $z = -t$, $0 \le t \le 1$.
C_3: $(1,3,-1)$ to $(1,3,0)$: $x = 1$, $y = 3$, $z = t-1$, $0 \le t \le 1$.
Then

$$\int_C yz\,dx + xz\,dy + xy\,dz$$
$$= 0 + \int_0^1 \left[\left(3t^2\right) + 3\left(t^2 - 2t\right) - 3\left(2t - t^2\right)\right] dt + \int_0^1 3\,dt$$
$$= \left[3t^3 - 6t^2\right]_0^1 + 3 = 0$$

15. $\mathbf{F}\left(\mathbf{r}\left(t\right)\right) = t^{10}\mathbf{i} - t^7\mathbf{j}$, $\mathbf{r}'\left(t\right) = 3t^2\mathbf{i} + 4t^3\mathbf{j}$.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left(3t^{12} - 4t^{10}\right) dt = \tfrac{3}{13} - \tfrac{4}{11} = -\tfrac{19}{143}$$

16.
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left\langle t^2 + t^4, -t^2, -4t^4\right\rangle \cdot \left\langle 1, 2t, 4t^3\right\rangle dt$$
$$= \int_0^1 \left(t^2 + t^4 - 2t^3 - 16t^7\right) dt$$
$$= \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{2} - 2 = -\tfrac{59}{30}$$

17.
$$\int_C \mathbf{F}\cdot d\mathbf{r}$$
$$= \int_0^{\pi/2} \left\langle \sin^2 t, \sin t\cos t, t^4\right\rangle \cdot \langle \cos t, -\sin t, 2t\rangle\, dt$$
$$= \int_0^{\pi/2} \left(\sin^2 t\cos t - \sin^2 t\cos t + 2t^5\right) dt$$
$$= \left[\tfrac{1}{3}t^6\right]_0^{\pi/2} = \frac{\pi^6}{192}$$

18. A calculator or CAS gives

$$\int_C x\sin y\,ds = \int_1^2 \ln t\sin\left(e^{-t}\right)\sqrt{(1/t)^2 + \left(-e^{-t}\right)^2}\,dt$$
$$\approx 0.052$$

19. A calculator or CAS gives

$$\int_C z^2\ln\left(1 + x^2 + y^2\right) ds$$
$$= \int_0^1 t^6\ln\left(1 + t^2 + t^4\right)\sqrt{1 + (2t)^2 + \left(3t^2\right)^2}\,dt \approx 0.396$$

20.
$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left\langle t^6, -t^5, -t^7\right\rangle \cdot \left\langle 2t, -3t^2, 4t^3\right\rangle dt$$
$$= \int_0^1 \left(5t^7 - 4t^{10}\right) dt = \tfrac{5}{8} - \tfrac{4}{11} = \tfrac{23}{88}$$