

**7.7** APPLIED PROJECT: WHICH IS FASTER, GOING UP OR COMING DOWN?

This project can be completed anytime after you have studied Section 7.7 in the textbook.

■ In modeling force due to air resistance, various functions have been used, depending on the physical characteristics and speed of the ball. Here we use a linear model,  $-pv$ , but a quadratic model ( $-pv^2$  on the way up and  $pv^2$  on the way down) is another possibility for higher speeds (see Exercise 48 in Section 7.7). For a golf ball, experiments have shown that a good model is  $-pv^{1.3}$  going up and  $p|v|^{1.3}$  coming down. But no matter which force function  $-f(v)$  is used [where  $f(v) > 0$  for  $v > 0$  and  $f(v) < 0$  for  $v < 0$ ], the answer to the question remains the same. See F. Brauer, "What Goes Up Must Come Down, Eventually," *Amer. Math. Monthly* 108 (2001), p. 437–440.

Suppose you throw a ball into the air. Do you think it takes longer to reach its maximum height or to fall back to Earth from its maximum height? We will solve the problem in this project but, before getting started, think about that situation and make a guess based on your physical intuition.

1. A ball with mass  $m$  is projected vertically upward from Earth's surface with a positive initial velocity  $v_0$ . We assume the forces acting on the ball are the force of gravity and a retarding force of air resistance with direction opposite to the direction of motion and with magnitude  $p|v(t)|$ , where  $p$  is a positive constant and  $v(t)$  is the velocity of the ball at time  $t$ . In both the ascent and the descent, the total force acting on the ball is  $-pv - mg$ . [During ascent,  $v(t)$  is positive and the resistance acts downward; during descent,  $v(t)$  is negative and the resistance acts upward.] So, by Newton's Second Law, the equation of motion is

$$mv' = -pv - mg$$

Solve this differential equation to show that the velocity is

$$v(t) = \left( v_0 + \frac{mg}{p} \right) e^{-pt/m} - \frac{mg}{p}$$

2. Show that the height of the ball, until it hits the ground, is

$$y(t) = \left( v_0 + \frac{mg}{p} \right) \frac{m}{p} (1 - e^{-pt/m}) - \frac{mgt}{p}$$

3. Let  $t_1$  be the time that the ball takes to reach its maximum height. Show that

$$t_1 = \frac{m}{p} \ln \left( \frac{mg + pv_0}{mg} \right)$$

Find this time for a ball with mass 1 kg and initial velocity 20 m/s. Assume the air resistance is  $\frac{1}{10}$  of the speed.

4. Let  $t_2$  be the time at which the ball falls back to Earth. For the particular ball in Problem 3, estimate  $t_2$  by using a graph of the height function  $y(t)$ . Which is faster, going up or coming down?
5. In general, it's not easy to find  $t_2$  because it's impossible to solve the equation  $y(t) = 0$  explicitly. We can, however, use an indirect method to determine whether ascent or descent is faster; we determine whether  $y(2t_1)$  is positive or negative. Show that

$$y(2t_1) = \frac{m^2g}{p^2} \left( x - \frac{1}{x} - 2 \ln x \right)$$

where  $x = e^{pt_1/m}$ . Then show that  $x > 1$  and the function

$$f(x) = x - \frac{1}{x} - 2 \ln x$$

is increasing for  $x > 1$ . Use this result to decide whether  $y(2t_1)$  is positive or negative. What can you conclude? Is ascent or descent faster?