APPLICATIONS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

Second-order linear differential equations have a variety of applications in science and engineering. In this section we explore two of them: the vibration of springs and electric circuits.

VIBRATING SPRINGS

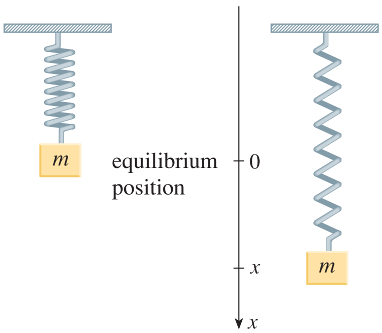

FIGURE 1

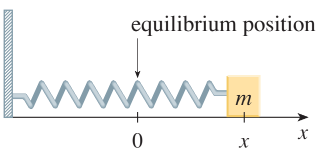

FIGURE 2

We consider the motion of an object with mass m at the end of a spring that is either vertical (as in Figure 1) or horizontal on a level surface (as in Figure 2).

In Section 7.6 we discussed Hooke's Law, which says that if the spring is stretched (or compressed) x units from its natural length, then it exerts a force that is proportional to x:

$$\text{restoring force} = -kx$$

where k is a positive constant (called the **spring constant**). If we ignore any external resisting forces (due to air resistance or friction) then, by Newton's Second Law (force equals mass times acceleration), we have

$$\boxed{1} \qquad m\frac{d^2x}{dt^2} = -kx \qquad \text{or} \qquad m\frac{d^2x}{dt^2} + kx = 0$$

This is a second-order linear differential equation. Its auxiliary equation is $mr^2 + k = 0$ with roots $r = \pm\omega i$, where $\omega = \sqrt{k/m}$. Thus, the general solution is

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

which can also be written as

$$x(t) = A\cos(\omega t + \delta)$$

where $\qquad \omega = \sqrt{k/m} \quad$ (frequency)

$$A = \sqrt{c_1^2 + c_2^2} \quad \text{(amplitude)}$$

$$\cos \delta = \frac{c_1}{A} \qquad \sin \delta = -\frac{c_2}{A} \qquad (\delta \text{ is the phase angle})$$

(See Exercise 17.) This type of motion is called **simple harmonic motion**.

EXAMPLE 1 A spring with a mass of 2 kg has natural length 0.5 m. A force of 25.6 N is required to maintain it stretched to a length of 0.7 m. If the spring is stretched to a length of 0.7 m and then released with initial velocity 0, find the position of the mass at any time t.

SOLUTION From Hooke's Law, the force required to stretch the spring is

$$k(0.2) = 25.6$$

so $k = 25.6/0.2 = 128$. Using this value of the spring constant k, together with $m = 2$ in Equation 1, we have

$$2\frac{d^2x}{dt^2} + 128x = 0$$

As in the earlier general discussion, the solution of this equation is

$$\boxed{2} \qquad x(t) = c_1 \cos 8t + c_2 \sin 8t$$

We are given the initial condition that $x(0) = 0.2$. But, from Equation 2, $x(0) = c_1$. Therefore, $c_1 = 0.2$. Differentiating Equation 2, we get

$$x'(t) = -8c_1 \sin 8t + 8c_2 \cos 8t$$

Since the initial velocity is given as $x'(0) = 0$, we have $c_2 = 0$ and so the solution is

$$x(t) = \tfrac{1}{5} \cos 8t$$

■

DAMPED VIBRATIONS

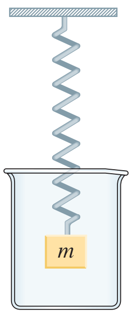

FIGURE 3

Schwinn Cycling and Fitness

We next consider the motion of a spring that is subject to a frictional force (in the case of the horizontal spring of Figure 2) or a damping force (in the case where a vertical spring moves through a fluid as in Figure 3). An example is the damping force supplied by a shock absorber in a car or a bicycle.

We assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion. (This has been confirmed, at least approximately, by some physical experiments.) Thus

$$\text{damping force} = -c\,\frac{dx}{dt}$$

where c is a positive constant, called the **damping constant**. Thus, in this case, Newton's Second Law gives

$$m\frac{d^2x}{dt^2} = \text{restoring force} + \text{damping force} = -kx - c\,\frac{dx}{dt}$$

or

3
$$\boxed{m\frac{d^2x}{dt^2} + c\,\frac{dx}{dt} + kx = 0}$$

Equation 3 is a second-order linear differential equation and its auxiliary equation is $mr^2 + cr + k = 0$. The roots are

4
$$r_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \qquad r_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}$$

We need to discuss three cases.

CASE I ▫ **$c^2 - 4mk > 0$ (overdamping)**
In this case r_1 and r_2 are distinct real roots and

$$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Since c, m, and k are all positive, we have $\sqrt{c^2 - 4mk} < c$, so the roots r_1 and r_2 given by Equations 4 must both be negative. This shows that $x \to 0$ as $t \to \infty$. Typical graphs of x as a function of t are shown in Figure 4. Notice that oscillations do not occur. (It's possible for the mass to pass through the equilibrium position once, but only once.) This is because $c^2 > 4mk$ means that there is a strong damping force (high-viscosity oil or grease) compared with a weak spring or small mass.

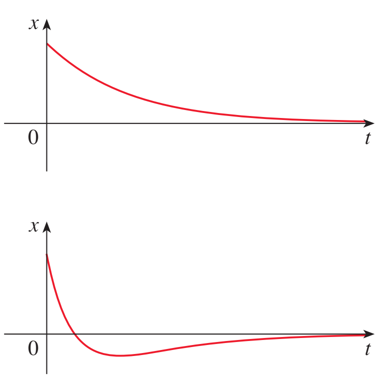

FIGURE 4
Overdamping

CASE II ▫ **$c^2 - 4mk = 0$ (critical damping)**
This case corresponds to equal roots

$$r_1 = r_2 = -\frac{c}{2m}$$

and the solution is given by

$$x = (c_1 + c_2 t)e^{-(c/2m)t}$$

It is similar to Case I, and typical graphs resemble those in Figure 4 (see Exercise 12), but the damping is just sufficient to suppress vibrations. Any decrease in the viscosity of the fluid leads to the vibrations of the following case.

CASE III ▫ $c^2 - 4mk < 0$ **(underdamping)**
Here the roots are complex:

$$\left.\begin{matrix} r_1 \\ r_2 \end{matrix}\right\} = -\frac{c}{2m} \pm \omega i$$

where
$$\omega = \frac{\sqrt{4mk - c^2}}{2m}$$

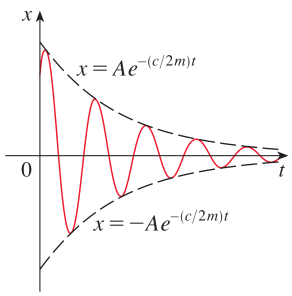

FIGURE 5
Underdamping

The solution is given by

$$x = e^{-(c/2m)t}(c_1 \cos \omega t + c_2 \sin \omega t)$$

We see that there are oscillations that are damped by the factor $e^{-(c/2m)t}$. Since $c > 0$ and $m > 0$, we have $-(c/2m) < 0$ so $e^{-(c/2m)t} \to 0$ as $t \to \infty$. This implies that $x \to 0$ as $t \to \infty$; that is, the motion decays to 0 as time increases. A typical graph is shown in Figure 5.

EXAMPLE 2 Suppose that the spring of Example 1 is immersed in a fluid with damping constant $c = 40$. Find the position of the mass at any time t if it starts from the equilibrium position and is given a push to start it with an initial velocity of 0.6 m/s.

SOLUTION From Example 1 the mass is $m = 2$ and the spring constant is $k = 128$, so the differential equation (3) becomes

$$2\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 128x = 0$$

or
$$\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 64x = 0$$

The auxiliary equation is $r^2 + 20r + 64 = (r + 4)(r + 16) = 0$ with roots -4 and -16, so the motion is overdamped and the solution is

$$x(t) = c_1 e^{-4t} + c_2 e^{-16t}$$

■■ Figure 6 shows the graph of the position function for the overdamped motion in Example 2.

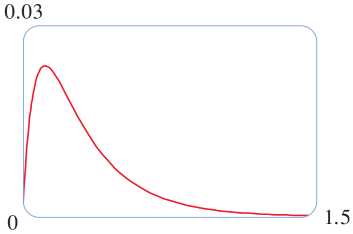

FIGURE 6

We are given that $x(0) = 0$, so $c_1 + c_2 = 0$. Differentiating, we get

$$x'(t) = -4c_1 e^{-4t} - 16c_2 e^{-16t}$$

so
$$x'(0) = -4c_1 - 16c_2 = 0.6$$

Since $c_2 = -c_1$, this gives $12c_1 = 0.6$ or $c_1 = 0.05$. Therefore

$$x = 0.05(e^{-4t} - e^{-16t})$$ ■

FORCED VIBRATIONS

Suppose that, in addition to the restoring force and the damping force, the motion of the spring is affected by an external force $F(t)$. Then Newton's Second Law gives

$$\begin{aligned} m\frac{d^2x}{dt^2} &= \text{restoring force} + \text{damping force} + \text{external force} \\ &= -kx - c\frac{dx}{dt} + F(t) \end{aligned}$$

Thus, instead of the homogeneous equation (3), the motion of the spring is now governed by the following nonhomogeneous differential equation:

5 $$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)}$$

The motion of the spring can be determined by the methods of *Additional Topics: Nonhomogeneous Linear Equations*.

A commonly occurring type of external force is a periodic force function

$$F(t) = F_0 \cos \omega_0 t \qquad \text{where} \quad \omega_0 \neq \omega = \sqrt{k/m}$$

In this case, and in the absence of a damping force ($c = 0$), you are asked in Exercise 9 to use the method of undetermined coefficients to show that

6 $$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t$$

If $\omega_0 = \omega$, then the applied frequency reinforces the natural frequency and the result is vibrations of large amplitude. This is the phenomenon of **resonance** (see Exercise 10).

ELECTRIC CIRCUITS

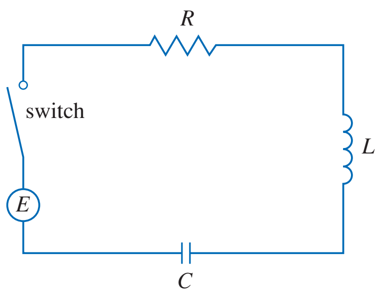

FIGURE 7

In *Additional Topics: Linear Differential Equations* we were able to use first-order linear equations to analyze electric circuits that contain a resistor and inductor. Now that we know how to solve second-order linear equations, we are in a position to analyze the circuit shown in Figure 7. It contains an electromotive force E (supplied by a battery or generator), a resistor R, an inductor L, and a capacitor C, in series. If the charge on the capacitor at time t is $Q = Q(t)$, then the current is the rate of change of Q with respect to t: $I = dQ/dt$. It is known from physics that the voltage drops across the resistor, inductor, and capacitor are

$$RI \qquad L\frac{dI}{dt} \qquad \frac{Q}{C}$$

respectively. Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage:

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t)$$

Since $I = dQ/dt$, this equation becomes

7
$$\boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)}$$

which is a second-order linear differential equation with constant coefficients. If the charge Q_0 and the current I_0 are known at time 0, then we have the initial conditions

$$Q(0) = Q_0 \qquad Q'(0) = I(0) = I_0$$

and the initial-value problem can be solved by the methods of *Additional Topics: Nonhomogeneous Linear Equations*.

A differential equation for the current can be obtained by differentiating Equation 7 with respect to t and remembering that $I = dQ/dt$:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t)$$

EXAMPLE 3 Find the charge and current at time t in the circuit of Figure 7 if $R = 40\ \Omega$, $L = 1$ H, $C = 16 \times 10^{-4}$ F, $E(t) = 100\cos 10t$, and the initial charge and current are both 0.

SOLUTION With the given values of L, R, C, and $E(t)$, Equation 7 becomes

8
$$\frac{d^2Q}{dt^2} + 40\frac{dQ}{dt} + 625Q = 100\cos 10t$$

The auxiliary equation is $r^2 + 40r + 625 = 0$ with roots

$$r = \frac{-40 \pm \sqrt{-900}}{2} = -20 \pm 15i$$

so the solution of the complementary equation is

$$Q_c(t) = e^{-20t}(c_1\cos 15t + c_2\sin 15t)$$

For the method of undetermined coefficients we try the particular solution

$$Q_p(t) = A\cos 10t + B\sin 10t$$

Then
$$Q_p'(t) = -10A\sin 10t + 10B\cos 10t$$

$$Q_p''(t) = -100A\cos 10t - 100B\sin 10t$$

Substituting into Equation 8, we have

$$(-100A\cos 10t - 100B\sin 10t) + 40(-10A\sin 10t + 10B\cos 10t)$$
$$+ 625(A\cos 10t + B\sin 10t) = 100\cos 10t$$

or
$$(525A + 400B)\cos 10t + (-400A + 525B)\sin 10t = 100\cos 10t$$

Equating coefficients, we have

$$\begin{aligned} 525A + 400B &= 100 \\ -400A + 525B &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} 21A + 16B &= 4 \\ -16A + 21B &= 0 \end{aligned}$$

The solution of this system is $A = \frac{84}{697}$ and $B = \frac{64}{697}$, so a particular solution is

$$Q_p(t) = \tfrac{1}{697}(84 \cos 10t + 64 \sin 10t)$$

and the general solution is

$$Q(t) = Q_c(t) + Q_p(t) = e^{-20t}(c_1 \cos 15t + c_2 \sin 15t) + \tfrac{4}{697}(21 \cos 10t + 16 \sin 10t)$$

Imposing the initial condition $Q(0) = 0$, we get

$$Q(0) = c_1 + \tfrac{84}{697} = 0 \qquad c_1 = -\tfrac{84}{697}$$

To impose the other initial condition we first differentiate to find the current:

$$I = \frac{dQ}{dt} = e^{-20t}[(-20c_1 + 15c_2) \cos 15t + (-15c_1 - 20c_2) \sin 15t]$$
$$+ \tfrac{40}{697}(-21 \sin 10t + 16 \cos 10t)$$

$$I(0) = -20c_1 + 15c_2 + \tfrac{640}{697} = 0 \qquad c_2 = -\tfrac{464}{2091}$$

Thus, the formula for the charge is

$$Q(t) = \frac{4}{697}\left[\frac{e^{-20t}}{3}(-63 \cos 15t - 116 \sin 15t) + (21 \cos 10t + 16 \sin 10t)\right]$$

and the expression for the current is

$$I(t) = \tfrac{1}{2091}[e^{-20t}(-1920 \cos 15t + 13{,}060 \sin 15t) + 120(-21 \sin 10t + 16 \cos 10t)]$$

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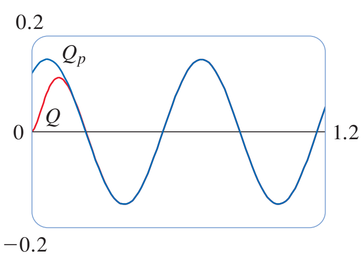


FIGURE 8

NOTE 1 ▫ In Example 3 the solution for $Q(t)$ consists of two parts. Since $e^{-20t} \to 0$ as $t \to \infty$ and both $\cos 15t$ and $\sin 15t$ are bounded functions,

$$Q_c(t) = \tfrac{4}{2091} e^{-20t}(-63 \cos 15t - 116 \sin 15t) \to 0 \qquad \text{as } t \to \infty$$

So, for large values of t,

$$Q(t) \approx Q_p(t) = \tfrac{4}{697}(21 \cos 10t + 16 \sin 10t)$$

and, for this reason, $Q_p(t)$ is called the **steady state solution**. Figure 8 shows how the graph of the steady state solution compares with the graph of Q in this case.

5 $$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

7 $$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

NOTE 2 ▫ Comparing Equations 5 and 7, we see that mathematically they are identical. This suggests the analogies given in the following chart between physical situations that, at first glance, are very different.

Spring system		Electric circuit	
x	displacement	Q	charge
dx/dt	velocity	$I = dQ/dt$	current
m	mass	L	inductance
c	damping constant	R	resistance
k	spring constant	$1/C$	elastance
$F(t)$	external force	$E(t)$	electromotive force

We can also transfer other ideas from one situation to the other. For instance, the steady state solution discussed in Note 1 makes sense in the spring system. And the phenomenon of resonance in the spring system can be usefully carried over to electric circuits as electrical resonance.

EXERCISES

A Click here for answers. S Click here for solutions.

1. A spring with a 3-kg mass is held stretched 0.6 m beyond its natural length by a force of 20 N. If the spring begins at its equilibrium position but a push gives it an initial velocity of 1.2 m/s, find the position of the mass after t seconds.

2. A spring with a 4-kg mass has natural length 1 m and is maintained stretched to a length of 1.3 m by a force of 24.3 N. If the spring is compressed to a length of 0.8 m and then released with zero velocity, find the position of the mass at any time t.

3. A spring with a mass of 2 kg has damping constant 14, and a force of 6 N is required to keep the spring stretched 0.5 m beyond its natural length. The spring is stretched 1 m beyond its natural length and then released with zero velocity. Find the position of the mass at any time t.

4. A spring with a mass of 3 kg has damping constant 30 and spring constant 123.
(a) Find the position of the mass at time t if it starts at the equilibrium position with a velocity of 2 m/s.
(b) Graph the position function of the mass.

5. For the spring in Exercise 3, find the mass that would produce critical damping.

6. For the spring in Exercise 4, find the damping constant that would produce critical damping.

7. A spring has a mass of 1 kg and its spring constant is $k = 100$. The spring is released at a point 0.1 m above its equilibrium position. Graph the position function for the following values of the damping constant c: 10, 15, 20, 25, 30. What type of damping occurs in each case?

8. A spring has a mass of 1 kg and its damping constant is $c = 10$. The spring starts from its equilibrium position with a velocity of 1 m/s. Graph the position function for the following values of the spring constant k: 10, 20, 25, 30, 40. What type of damping occurs in each case?

9. Suppose a spring has mass m and spring constant k and let $\omega = \sqrt{k/m}$. Suppose that the damping constant is so small that the damping force is negligible. If an external force $F(t) = F_0 \cos \omega_0 t$ is applied, where $\omega_0 \neq \omega$, use the method of undetermined coefficients to show that the motion of the mass is described by Equation 6.

10. As in Exercise 9, consider a spring with mass m, spring constant k, and damping constant $c = 0$, and let $\omega = \sqrt{k/m}$. If an external force $F(t) = F_0 \cos \omega t$ is applied (the applied frequency equals the natural frequency), use the method of undetermined coefficients to show that the motion of the mass is given by $x(t) = c_1 \cos \omega t + c_2 \sin \omega t + (F_0/(2m\omega))t \sin \omega t$.

11. Show that if $\omega_0 \neq \omega$, but ω/ω_0 is a rational number, then the motion described by Equation 6 is periodic.

12. Consider a spring subject to a frictional or damping force.
(a) In the critically damped case, the motion is given by $x = c_1 e^{rt} + c_2 t e^{rt}$. Show that the graph of x crosses the t-axis whenever c_1 and c_2 have opposite signs.
(b) In the overdamped case, the motion is given by $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, where $r_1 > r_2$. Determine a condition on the relative magnitudes of c_1 and c_2 under which the graph of x crosses the t-axis at a positive value of t.

13. A series circuit consists of a resistor with $R = 20\ \Omega$, an inductor with $L = 1$ H, a capacitor with $C = 0.002$ F, and a 12-V battery. If the initial charge and current are both 0, find the charge and current at time t.

14. A series circuit contains a resistor with $R = 24\ \Omega$, an inductor with $L = 2$ H, a capacitor with $C = 0.005$ F, and a 12-V battery. The initial charge is $Q = 0.001$ C and the initial current is 0.
(a) Find the charge and current at time t.
(b) Graph the charge and current functions.

15. The battery in Exercise 13 is replaced by a generator producing a voltage of $E(t) = 12 \sin 10t$. Find the charge at time t.

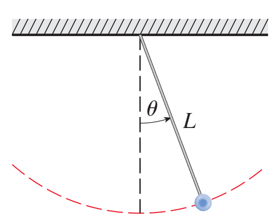
16. The battery in Exercise 14 is replaced by a generator producing a voltage of $E(t) = 12 \sin 10t$.
(a) Find the charge at time t.
(b) Graph the charge function.

17. Verify that the solution to Equation 1 can be written in the form $x(t) = A \cos(\omega t + \delta)$.

18. The figure shows a pendulum with length L and the angle θ from the vertical to the pendulum. It can be shown that θ, as a function of time, satisfies the nonlinear differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} \sin \theta = 0$$

where g is the acceleration due to gravity. For small values of θ we can use the linear approximation $\sin \theta \approx \theta$ and then the differential equation becomes linear.
(a) Find the equation of motion of a pendulum with length 1 m if θ is initially 0.2 rad and the initial angular velocity is $d\theta/dt = 1$ rad/s.
(b) What is the maximum angle from the vertical?
(c) What is the period of the pendulum (that is, the time to complete one back-and-forth swing)?
(d) When will the pendulum first be vertical?
(e) What is the angular velocity when the pendulum is vertical?

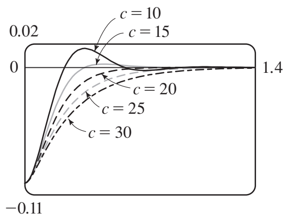


ANSWERS

S Click here for solutions.

1. $x = 0.36 \sin(10t/3)$ **3.** $x = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$ **5.** $\frac{49}{12}$ kg

7.



13. $Q(t) = (-e^{-10t}/250)(6 \cos 20t + 3 \sin 20t) + \frac{3}{125}$,
$I(t) = \frac{3}{5}e^{-10t} \sin 20t$

15. $Q(t) = e^{-10t}\left[\frac{3}{250} \cos 20t - \frac{3}{500} \sin 20t\right]$
$- \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t$

SOLUTIONS

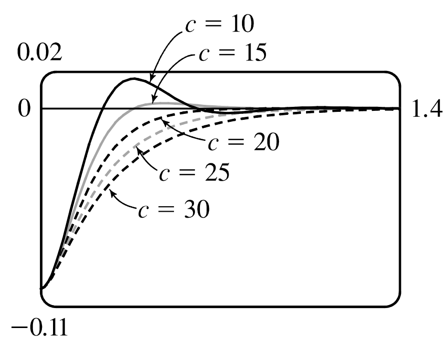
1. By Hooke's Law $k(0.6) = 20$ so $k = \frac{100}{3}$ is the spring constant and the differential equation is $3x'' + \frac{100}{3}x = 0$. The general solution is $x(t) = c_1 \cos\left(\frac{10}{3}t\right) + c_2 \sin\left(\frac{10}{3}t\right)$. But $0 = x(0) = c_1$ and $1.2 = x'(0) = \frac{10}{3}c_2$, so the position of the mass after t seconds is $x(t) = 0.36 \sin\left(\frac{10}{3}t\right)$.

3. $k(0.5) = 6$ or $k = 12$ is the spring constant, so the initial-value problem is $2x'' + 14x' + 12x = 0$, $x(0) = 1$, $x'(0) = 0$. The general solution is $x(t) = c_1 e^{-6t} + c_2 e^{-t}$. But $1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -6c_1 - c_2$. Thus the position is given by $x(t) = -\frac{1}{5}e^{-6t} + \frac{6}{5}e^{-t}$.

5. For critical damping we need $c^2 - 4mk = 0$ or $m = c^2/(4k) = 14^2/(4 \cdot 12) = \frac{49}{12}$ kg.

7. We are given $m = 1$, $k = 100$, $x(0) = -0.1$ and $x'(0) = 0$. From (3), the differential equation is $\frac{d^2x}{dt^2} + c\frac{dx}{dt} + 100x = 0$ with auxiliary equation $r^2 + cr + 100 = 0$. If $c = 10$, we have two complex roots $r = -5 \pm 5\sqrt{3}i$, so the motion is underdamped and the solution is $x = e^{-5t}\left[c_1 \cos\left(5\sqrt{3}\,t\right) + c_2 \sin\left(5\sqrt{3}\,t\right)\right]$. Then $-0.1 = x(0) = c_1$ and $0 = x'(0) = 5\sqrt{3}\,c_2 - 5c_1 \quad\Rightarrow\quad c_2 = -\frac{1}{10\sqrt{3}}$, so $x = e^{-5t}\left[-0.1\cos\left(5\sqrt{3}\,t\right) - \frac{1}{10\sqrt{3}}\sin\left(5\sqrt{3}\,t\right)\right]$. If $c = 15$, we again have underdamping since the auxiliary equation has roots $r = -\frac{15}{2} \pm \frac{5\sqrt{7}}{2}i$. The general solution is $x = e^{-15t/2}\left[c_1 \cos\left(\frac{5\sqrt{7}}{2}t\right) + c_2 \sin\left(\frac{5\sqrt{7}}{2}t\right)\right]$, so $-0.1 = x\,(0) = c_1$ and $0 = x'(0) = \frac{5\sqrt{7}}{2}c_2 - \frac{15}{2}c_1 \quad\Rightarrow\quad c_2 = -\frac{3}{10\sqrt{7}}$. Thus $x = e^{-15t/2}\left[-0.1\cos\left(\frac{5\sqrt{7}}{2}t\right) - \frac{3}{10\sqrt{7}}\sin\left(\frac{5\sqrt{7}}{2}t\right)\right]$. For $c = 20$, we have equal roots $r_1 = r_2 = -10$, so the oscillation is critically damped and the solution is $x = (c_1 + c_2 t)e^{-10t}$. Then $-0.1 = x(0) = c_1$ and $0 = x'(0) = -10c_1 + c_2 \quad\Rightarrow\quad c_2 = -1$, so $x = (-0.1 - t)e^{-10t}$. If $c = 25$ the auxiliary equation has roots $r_1 = -5$, $r_2 = -20$, so we have overdamping and the solution is $x = c_1 e^{-5t} + c_2 e^{-20t}$. Then $-0.1 = x(0) = c_1 + c_2$ and $0 = x'(0) = -5c_1 - 20c_2 \quad\Rightarrow\quad c_1 = -\frac{2}{15}$ and $c_2 = \frac{1}{30}$, so $x = -\frac{2}{15}e^{-5t} + \frac{1}{30}e^{-20t}$. If $c = 30$ we have roots $r = -15 \pm 5\sqrt{5}$, so the motion is overdamped and the solution is $x = c_1 e^{\left(-15+5\sqrt{5}\right)t} + c_2 e^{\left(-15-5\sqrt{5}\right)t}$. Then $-0.1 = x(0) = c_1 + c_2$ and $0 = x'(0) = \left(-15 + 5\sqrt{5}\right)c_1 + \left(-15 - 5\sqrt{5}\right)c_2 \quad\Rightarrow$ $c_1 = \frac{-5-3\sqrt{5}}{100}$ and $c_2 = \frac{-5+3\sqrt{5}}{100}$, so

$$x = \left(\frac{-5-3\sqrt{5}}{100}\right)e^{\left(-15+5\sqrt{5}\right)t} + \left(\frac{-5+3\sqrt{5}}{100}\right)e^{\left(-15-5\sqrt{5}\right)t}.$$



9. The differential equation is $mx'' + kx = F_0 \cos\omega_0 t$ and $\omega_0 \neq \omega = \sqrt{k/m}$. Here the auxiliary equation is $mr^2 + k = 0$ with roots $\pm\sqrt{k/m}\,i = \pm\omega i$ so $x_c(t) = c_1 \cos\omega t + c_2 \sin\omega t$. Since $\omega_0 \neq \omega$, try $x_p(t) = A\cos\omega_0 t + B\sin\omega_0 t$. Then we need $(m)\left(-\omega_0^2\right)\left(A\cos\omega_0 t + B\sin\omega_0 t\right) + k(A\cos\omega_0 t + B\sin\omega_0 t) = F_0\cos\omega_0 t$ or $A\left(k - m\omega_0^2\right) = F_0$ and $B\left(k - m\omega_0^2\right) = 0$. Hence $B = 0$ and $A = \frac{F_0}{k - m\omega_0^2} = \frac{F_0}{m(\omega^2 - \omega_0^2)}$ since $\omega^2 = \frac{k}{m}$. Thus the motion of the mass is given by $x(t) = c_1\cos\omega t + c_2\sin\omega t + \frac{F_0}{m(\omega^2 - \omega_0^2)}\cos\omega_0 t$.

11. From Equation 6, $x(t) = f(t) + g(t)$ where $f(t) = c_1 \cos \omega t + c_2 \sin \omega t$ and $g(t) = \dfrac{F_0}{m(\omega^2 - \omega_0^2)} \cos \omega_0 t$. Then f is periodic, with period $\frac{2\pi}{\omega}$, and if $\omega \neq \omega_0$, g is periodic with period $\frac{2\pi}{\omega_0}$. If $\frac{\omega}{\omega_0}$ is a rational number, then we can say $\frac{\omega}{\omega_0} = \frac{a}{b} \quad \Rightarrow \quad a = \frac{b\omega}{\omega_0}$ where a and b are non-zero integers. Then

$$\begin{aligned} x\left(t + a \cdot \tfrac{2\pi}{\omega}\right) &= f\left(t + a \cdot \tfrac{2\pi}{\omega}\right) + g\left(t + a \cdot \tfrac{2\pi}{\omega}\right) = f(t) + g\left(t + \tfrac{b\omega}{\omega_0} \cdot \tfrac{2\pi}{\omega}\right) \\ &= f(t) + g\left(t + b \cdot \tfrac{2\pi}{\omega_0}\right) = f(t) + g(t) = x(t) \end{aligned}$$

so $x(t)$ is periodic.

13. Here the initial-value problem for the charge is $Q'' + 20Q' + 500Q = 12$, $Q(0) = Q'(0) = 0$. Then $Q_c(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t)$ and try $Q_p(t) = A \quad \Rightarrow \quad 500A = 12$ or $A = \frac{3}{125}$.
The general solution is $Q(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) + \frac{3}{125}$. But $0 = Q(0) = c_1 + \frac{3}{125}$ and $Q'(t) = I(t) = e^{-10t}[(-10c_1 + 20c_2) \cos 20t + (-10c_2 - 20c_1) \sin 20t]$ but $0 = Q'(0) = -10c_1 + 20c_2$. Thus the charge is $Q(t) = -\frac{1}{250} e^{-10t}(6 \cos 20t + 3 \sin 20t) + \frac{3}{125}$ and the current is $I(t) = e^{-10t}\left(\frac{3}{5}\right) \sin 20t$.

15. As in Exercise 13, $Q_c(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t)$ but $E(t) = 12 \sin 10t$ so try $Q_p(t) = A \cos 10t + B \sin 10t$. Substituting into the differential equation gives $(-100A + 200B + 500A) \cos 10t + (-100B - 200A + 500B) \sin 10t = 12 \sin 10t \quad \Rightarrow \quad 400A + 200B = 0$ and $400B - 200A = 12$. Thus $A = -\frac{3}{250}$, $B = \frac{3}{125}$ and the general solution is $Q(t) = e^{-10t}(c_1 \cos 20t + c_2 \sin 20t) - \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t$. But $0 = Q(0) = c_1 - \frac{3}{250}$ so $c_1 = \frac{3}{250}$.
Also $Q'(t) = \frac{3}{25} \sin 10t + \frac{6}{25} \cos 10t + e^{-10t}[(-10c_1 + 20c_2) \cos 20t + (-10c_2 - 20c_1) \sin 20t]$ and $0 = Q'(0) = \frac{6}{25} - 10c_1 + 20c_2$ so $c_2 = -\frac{3}{500}$. Hence the charge is given by $Q(t) = e^{-10t}\left[\frac{3}{250} \cos 20t - \frac{3}{500} \sin 20t\right] - \frac{3}{250} \cos 10t + \frac{3}{125} \sin 10t$.

17. $x(t) = A \cos(\omega t + \delta) \quad \Leftrightarrow \quad x(t) = A[\cos \omega t \cos \delta - \sin \omega t \sin \delta] \quad \Leftrightarrow \quad x(t) = A\left(\dfrac{c_1}{A} \cos \omega t + \dfrac{c_2}{A} \sin \omega t\right)$ where $\cos \delta = c_1/A$ and $\sin \delta = -c_2/A \quad \Leftrightarrow \quad x(t) = c_1 \cos \omega t + c_2 \sin \omega t$. (Note that $\cos^2 \delta + \sin^2 \delta = 1 \quad \Rightarrow \quad c_1^2 + c_2^2 = A^2$.)