CHALLENGE PROBLEMS

CHAPTER 4

A Click here for answers. S Click here for solutions.

1. Show that $|\sin x - \cos x| \leqslant \sqrt{2}$ for all x.

2. Show that $x^2y^2(4 - x^2)(4 - y^2) \leqslant 16$ for all numbers x and y such that $|x| \leqslant 2$ and $|y| \leqslant 2$.

3. Let a and b be positive numbers. Show that not both of the numbers $a(1 - b)$ and $b(1 - a)$ can be greater than $\frac{1}{4}$.

4. Find the point on the parabola $y = 1 - x^2$ at which the tangent line cuts from the first quadrant the triangle with the smallest area.

5. Find the highest and lowest points on the curve $x^2 + xy + y^2 = 12$.

6. Water is flowing at a constant rate into a spherical tank. Let $V(t)$ be the volume of water in the tank and $H(t)$ be the height of the water in the tank at time t.
(a) What are the meanings of $V'(t)$ and $H'(t)$? Are these derivatives positive, negative, or zero?
(b) Is $V''(t)$ positive, negative, or zero? Explain.
(c) Let t_1, t_2, and t_3 be the times when the tank is one-quarter full, half full, and three-quarters full, respectively. Are the values $H''(t_1)$, $H''(t_2)$, and $H''(t_3)$ positive, negative, or zero? Why?

7. Find the absolute maximum value of the function

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - 2|}$$

8. Find a function f such that $f'(-1) = \frac{1}{2}$, $f'(0) = 0$, and $f''(x) > 0$ for all x, or prove that such a function cannot exist.

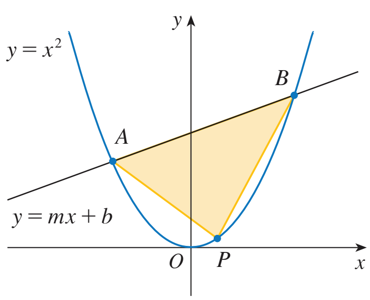


FIGURE FOR PROBLEM 9

9. The line $y = mx + b$ intersects the parabola $y = x^2$ in points A and B (see the figure). Find the point P on the arc AOB of the parabola that maximizes the area of the triangle PAB.

10. Sketch the graph of a function f such that $f'(x) < 0$ for all x, $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and $\lim_{x \to \pm\infty} [f(x) + x] = 0$.

11. Determine the values of the number a for which the function f has no critical number:

$$f(x) = (a^2 + a - 6)\cos 2x + (a - 2)x + \cos 1$$

12. Sketch the region in the plane consisting of all points (x, y) such that

$$2xy \leqslant |x - y| \leqslant x^2 + y^2$$

13. Let ABC be a triangle with $\angle BAC = 120°$ and $|AB| \cdot |AC| = 1$.
(a) Express the length of the angle bisector AD in terms of $x = |AB|$.
(b) Find the largest possible value of $|AD|$.

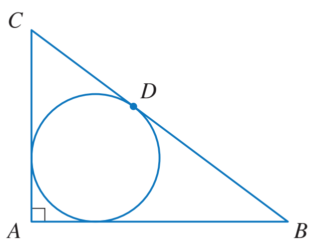


FIGURE FOR PROBLEM 14

14. (a) Let ABC be a triangle with right angle A and hypotenuse $a = |BC|$. (See the figure.) If the inscribed circle touches the hypotenuse at D, show that

$$|CD| = \tfrac{1}{2}(|BC| + |AC| - |AB|)$$

(b) If $\theta = \frac{1}{2}\angle C$, express the radius r of the inscribed circle in terms of a and θ.
(c) If a is fixed and θ varies, find the maximum value of r.

15. A triangle with sides a, b, and c varies with time t, but its area never changes. Let θ be the angle opposite the side of length a and suppose θ always remains acute.
(a) Express $d\theta/dt$ in terms of b, c, θ, db/dt, and dc/dt.
(b) Express da/dt in terms of the quantities in part (a).

16. $ABCD$ is a square piece of paper with sides of length 1 m. A quarter-circle is drawn from B to D with center A. The piece of paper is folded along EF, with E on AB and F on AD, so that A falls on the quarter-circle. Determine the maximum and minimum areas that the triangle AEF could have.

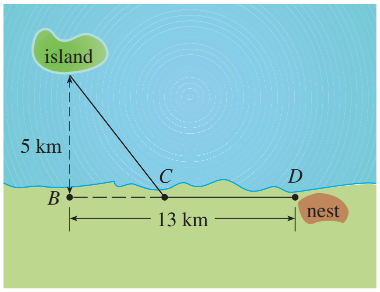


FIGURE FOR PROBLEM 17

17. Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours. It is believed that more energy is required to fly over water than land because air generally rises over land and falls over water during the day. A bird with these tendencies is released from an island that is 5 km from the nearest point B on a straight shoreline, flies to a point C on the shoreline, and then flies along the shoreline to its nesting area D. Assume that the bird instinctively chooses a path that will minimize its energy expenditure. Points B and D are 13 km apart.

(a) In general, if it takes 1.4 times as much energy to fly over water as land, to what point C should the bird fly in order to minimize the total energy expended in returning to its nesting area?

(b) Let W and L denote the energy (in joules) per kilometer flown over water and land, respectively. What would a large value of the ratio W/L mean in terms of the bird's flight? What would a small value mean? Determine the ratio W/L corresponding to the minimum expenditure of energy.

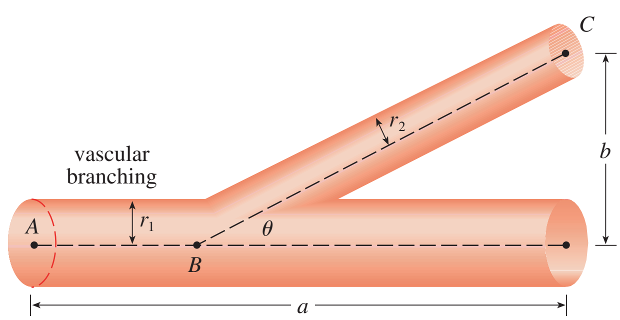
(c) What should the value of W/L be in order for the bird to fly directly to its nesting area D? What should the value of W/L be for the bird to fly to B and then along the shore to D?

(d) If the ornithologists observe that birds of a certain species reach the shore at a point 4 km from B, how many times more energy does it take a bird to fly over water than land?

18. The blood vascular system consists of blood vessels (arteries, arterioles, capillaries, and veins) that convey blood from the heart to the organs and back to the heart. This system should work so as to minimize the energy expended by the heart in pumping the blood. In particular, this energy is reduced when the resistance of the blood is lowered. One of Poiseuille's Laws gives the resistance R of the blood as

$$R = C\frac{L}{r^4}$$

where L is the length of the blood vessel, r is the radius, and C is a positive constant determined by the viscosity of the blood. (Poiseuille established this law experimentally.) The figure shows a main blood vessel with radius r_1 branching at an angle θ into a smaller vessel with radius r_2.



(a) Use Poiseuille's Law to show that the total resistance of the blood along the path ABC is

$$R = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right)$$

where a and b are the distances shown in the figure.

(b) Prove that this resistance is minimized when

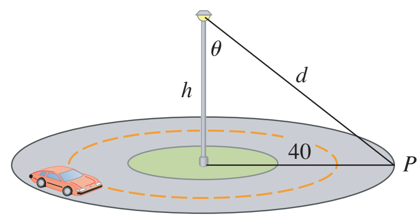
$$\cos\theta = \frac{r_2^4}{r_1^4}$$

(c) Find the optimal branching angle (correct to the nearest degree) when the radius of the smaller blood vessel is two-thirds the radius of the larger vessel.

19. A light is to be placed atop a pole of height h feet to illuminate a busy traffic circle, which has a radius of 40 ft. The intensity of illumination I at any point P on the circle is directly proportional to the cosine of the angle θ (see the figure) and inversely proportional to the square of the distance d from the source.

(a) How tall should the light pole be to maximize I?

(b) Suppose that the light pole is h feet tall and that a woman is walking away from the base of the pole at the rate of 4 ft/s. At what rate is the intensity of the light at the point on her back 4 ft above the ground decreasing when she reaches the outer edge of the traffic circle?



20. If a projectile is fired with an initial velocity v at an angle of inclination θ from the horizontal, then its trajectory, neglecting air resistance, is the parabola

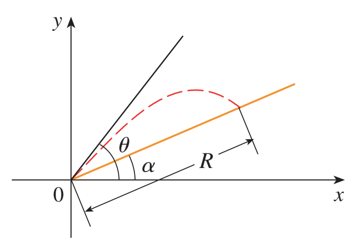
$$y = (\tan\theta)x - \frac{g}{2v^2\cos^2\theta}x^2 \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

(a) Suppose the projectile is fired from the base of a plane that is inclined at an angle α, $\alpha > 0$, from the horizontal, as shown in the figure. Show that the range of the projectile, measured up the slope, is given by

$$R(\theta) = \frac{2v^2\cos\theta\,\sin(\theta - \alpha)}{g\cos^2\alpha}$$

(b) Determine θ so that R is a maximum.

(c) Suppose the plane is at an angle α *below* the horizontal. Determine the range R in this case, and determine the angle at which the projectile should be fired to maximize R.

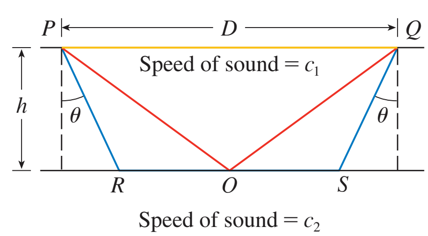


21. The speeds of sound c_1 in an upper layer and c_2 in a lower layer of rock and the thickness h of the upper layer can be determined by seismic exploration if the speed of sound in the lower layer is greater than the speed in the upper layer. A dynamite charge is detonated at a point P and the transmitted signals are recorded at a point Q, which is a distance D from P. The first signal to arrive at Q travels along the surface and takes T_1 seconds. The next signal travels from P to a point R, from R to S in the lower layer, and then to Q, taking T_2 seconds. The third signal is reflected off the lower layer at the midpoint O of RS and takes T_3 seconds to reach Q.

(a) Express T_1, T_2, and T_3 in terms of D, h, c_1, c_2, and θ.

(b) Show that T_2 is a minimum when $\sin\theta = c_1/c_2$.

(c) Suppose that $D = 1$ km, $T_1 = 0.26$ s, $T_2 = 0.32$ s, $T_3 = 0.34$ s. Find c_1, c_2, and h.



Note: Geophysicists use this technique when studying the structure of Earth's crust, whether searching for oil or examining fault lines.

22. For what values of c is there a straight line that intersects the curve

$$y = x^4 + cx^3 + 12x^2 - 5x + 2$$

in four distinct points?

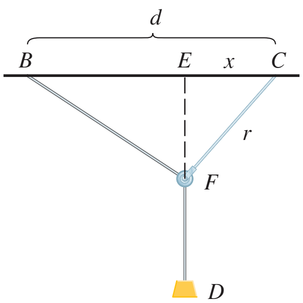


FIGURE FOR PROBLEM 23

23. One of the problems posed by the Marquis de l'Hospital in his calculus textbook *Analyse des Infiniment Petits* concerns a pulley that is attached to the ceiling of a room at a point C by a rope of length r. At another point B on the ceiling, at a distance d from C (where $d > r$), a rope of length ℓ is attached and passed through the pulley at F and connected to a weight W. The weight is released and comes to rest at its equilibrium position D. As l'Hospital argued, this happens when the distance $|ED|$ is maximized. Show that when the system reaches equilibrium, the value of x is

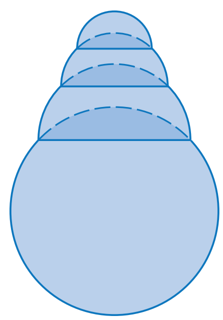
$$\frac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right)$$

Notice that this expression is independent of both W and ℓ.

24. Given a sphere with radius r, find the height of a pyramid of minimum volume whose base is a square and whose base and triangular faces are all tangent to the sphere. What if the base of the pyramid is a regular n-gon (a polygon with n equal sides and angles)? (Use the fact that the volume of a pyramid is $\frac{1}{3}Ah$, where A is the area of the base.)

25. Assume that a snowball melts so that its volume decreases at a rate proportional to its surface area. If it takes three hours for the snowball to decrease to half its original volume, how much longer will it take for the snowball to melt completely?

26. A hemispherical bubble is placed on a spherical bubble of radius 1. A smaller hemispherical bubble is then placed on the first one. This process is continued until n chambers, including the sphere, are formed. (The figure shows the case $n = 4$.) Use mathematical induction to prove that the maximum height of any bubble tower with n chambers is $1 + \sqrt{n}$.



ANSWERS

S Solutions

5. $(-2, 4), (2, -4)$ **7.** $\frac{4}{3}$ **9.** $(m/2, m^2/4)$ **11.** $-3.5 < a < -2.5$ **13.** (a) $x/(x^2 + 1)$ (b) $\frac{1}{2}$

15. (a) $-\tan\theta\left[\frac{1}{c}\frac{dc}{dt} + \frac{1}{b}\frac{db}{dt}\right]$ (b) $\dfrac{b\dfrac{db}{dt} + c\dfrac{dc}{dt} - \left(b\dfrac{dc}{dt} + c\dfrac{db}{dt}\right)\sec\theta}{\sqrt{b^2 + c^2 - 2bc\cos\theta}}$

17. (a) About 5.1 km from B (b) C is close to B; C is close to D; $W/L = \sqrt{25 + x^2}/x$, where $x = |BC|$
(c) ≈ 1.07; no such value (d) $\sqrt{41}/4 \approx 1.6$

19. (a) $20\sqrt{2} \approx 28$ ft (b) $\dfrac{dI}{dt} = \dfrac{-480k(h-4)}{[(h-4)^2 + 1600]^{5/2}}$, where k is the constant of proportionality

21. (a) $T_1 = D/c_1$, $T_2 = (2h\sec\theta)/c_1 + (D - 2h\tan\theta)/c_2$, $T_3 = \sqrt{4h^2 + D^2}/c_1$
(c) $c_1 \approx 3.85$ km/s, $c_2 \approx 7.66$ km/s, $h \approx 0.42$ km

25. $3/(\sqrt[3]{2} - 1) \approx 11\frac{1}{2}$ h

SOLUTIONS

E Exercises

1. Let $f(x) = \sin x - \cos x$ on $[0, 2\pi]$ since f has period 2π. $f'(x) = \cos x + \sin x = 0 \quad \Leftrightarrow \quad \cos x = -\sin x \quad \Leftrightarrow$ $\tan x = -1 \quad \Leftrightarrow \quad x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. Evaluating f at its critical numbers and endpoints, we get $f(0) = -1$, $f\left(\frac{3\pi}{4}\right) = \sqrt{2}$, $f\left(\frac{7\pi}{4}\right) = -\sqrt{2}$, and $f(2\pi) = -1$. So f has absolute maximum value $\sqrt{2}$ and absolute minimum value $-\sqrt{2}$. Thus, $-\sqrt{2} \le \sin x - \cos x \le \sqrt{2} \quad \Rightarrow \quad |\sin x - \cos x| \le \sqrt{2}$.

3. First we show that $x(1-x) \le \frac{1}{4}$ for all x. Let $f(x) = x(1-x) = x - x^2$. Then $f'(x) = 1 - 2x$. This is 0 when $x = \frac{1}{2}$ and $f'(x) > 0$ for $x < \frac{1}{2}$, $f'(x) < 0$ for $x > \frac{1}{2}$, so the absolute maximum of f is $f\left(\frac{1}{2}\right) = \frac{1}{4}$. Thus, $x(1-x) \le \frac{1}{4}$ for all x.

Now suppose that the given assertion is false, that is, $a(1-b) > \frac{1}{4}$ and $b(1-a) > \frac{1}{4}$. Multiply these inequalities: $a(1-b)b(1-a) > \frac{1}{16} \quad \Rightarrow \quad [a(1-a)]\,[b(1-b)] > \frac{1}{16}$. But we know that $a(1-a) \le \frac{1}{4}$ and $b(1-b) \le \frac{1}{4} \quad \Rightarrow \quad [a(1-a)]\,[b(1-b)] \le \frac{1}{16}$. Thus, we have a contradiction, so the given assertion is proved.

5. Differentiating $x^2 + xy + y^2 = 12$ implicitly with respect to x gives $2x + y + x\,\dfrac{dy}{dx} + 2y\,\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$. At a highest or lowest point, $\dfrac{dy}{dx} = 0 \quad \Leftrightarrow \quad y = -2x$. Substituting $-2x$ for y in the original equation gives $x^2 + x(-2x) + (-2x)^2 = 12$, so $3x^2 = 12$ and $x = \pm 2$. If $x = 2$, then $y = -2x = -4$, and if $x = -2$ then $y = 4$. Thus, the highest and lowest points are $(-2, 4)$ and $(2, -4)$.

7. $f(x) = \dfrac{1}{1+|x|} + \dfrac{1}{1+|x-2|}$

$$= \begin{cases} \dfrac{1}{1-x} + \dfrac{1}{1-(x-2)} & \text{if } x < 0 \\[2mm] \dfrac{1}{1+x} + \dfrac{1}{1-(x-2)} & \text{if } 0 \le x < 2 \\[2mm] \dfrac{1}{1+x} + \dfrac{1}{1+(x-2)} & \text{if } x \ge 2 \end{cases} \quad \Rightarrow \quad f'(x) = \begin{cases} \dfrac{1}{(1-x)^2} + \dfrac{1}{(3-x)^2} & \text{if } x < 0 \\[2mm] \dfrac{-1}{(1+x)^2} + \dfrac{1}{(3-x)^2} & \text{if } 0 < x < 2 \\[2mm] \dfrac{-1}{(1+x)^2} - \dfrac{1}{(x-1)^2} & \text{if } x > 2 \end{cases}$$

We see that $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 2$. For $0 < x < 2$, we have

$$f'(x) = \frac{1}{(3-x)^2} - \frac{1}{(x+1)^2} = \frac{\left(x^2+2x+1\right) - \left(x^2-6x+9\right)}{(3-x)^2(x+1)^2} = \frac{8\,(x-1)}{(3-x)^2(x+1)^2},$$ so $f'(x) < 0$ for $0 < x < 1$, $f'(1) = 0$ and $f'(x) > 0$ for $1 < x < 2$. We have shown that $f'(x) > 0$ for $x < 0$; $f'(x) < 0$ for $0 < x < 1$; $f'(x) > 0$ for $1 < x < 2$; and $f'(x) < 0$ for $x > 2$. Therefore, by the First Derivative Test, the local maxima of f are at $x = 0$ and $x = 2$, where f takes the value $\frac{4}{3}$. Therefore, $\frac{4}{3}$ is the absolute maximum value of f.

9. $A = \left(x_1, x_1^2\right)$ and $B = \left(x_2, x_2^2\right)$, where x_1 and x_2 are the solutions of the quadratic equation $x^2 = mx + b$. Let $P = \left(x, x^2\right)$ and set $A_1 = (x_1, 0)$, $B_1 = (x_2, 0)$, and $P_1 = (x, 0)$. Let $f(x)$ denote the area of triangle PAB. Then $f(x)$ can be expressed in terms of the areas of three trapezoids as follows:

$$\begin{aligned} f(x) &= \text{area}\,(A_1ABB_1) - \text{area}\,(A_1APP_1) - \text{area}\,(B_1BPP_1) \\ &= \tfrac{1}{2}\left(x_1^2 + x_2^2\right)(x_2 - x_1) - \tfrac{1}{2}\left(x_1^2 + x^2\right)(x - x_1) - \tfrac{1}{2}\left(x^2 + x_2^2\right)(x_2 - x) \end{aligned}$$

After expanding and canceling terms, we get

$f(x) = \frac{1}{2}\left(x_2x_1^2 - x_1x_2^2 - xx_1^2 + x_1x^2 - x_2x^2 + xx_2^2\right) = \frac{1}{2}\left[x_1^2(x_2 - x) + x_2^2(x - x_1) + x^2(x_1 - x_2)\right]$

$f'(x) = \frac{1}{2}\left[-x_1^2 + x_2^2 + 2x(x_1 - x_2)\right]$. $f''(x) = \frac{1}{2}[2(x_1 - x_2)] = x_1 - x_2 < 0$ since $x_2 > x_1$.

$f'(x) = 0 \quad\Rightarrow\quad 2x(x_1 - x_2) = x_1^2 - x_2^2 \quad\Rightarrow\quad x_P = \frac{1}{2}(x_1 + x_2)$.

$$\begin{aligned} f(x_P) &= \tfrac{1}{2}\left(x_1^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + x_2^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + \tfrac{1}{4}(x_1 + x_2)^2(x_1 - x_2)\right) \\ &= \tfrac{1}{2}\left[\tfrac{1}{2}(x_2 - x_1)\left(x_1^2 + x_2^2\right) - \tfrac{1}{4}(x_2 - x_1)(x_1 + x_2)^2\right] \\ &= \tfrac{1}{8}(x_2 - x_1)\left[2\left(x_1^2 + x_2^2\right) - \left(x_1^2 + 2x_1x_2 + x_2^2\right)\right] \\ &= \tfrac{1}{8}(x_2 - x_1)(x_1 - x_2)^2 = \tfrac{1}{8}(x_2 - x_1)(x_2 - x_1)^2 = \tfrac{1}{8}(x_2 - x_1)^3 \end{aligned}$$

To put this in terms of m and b, we solve the system $y = x_1^2$ and $y = mx_1 + b$, giving us $x_1^2 - mx_1 - b = 0 \quad\Rightarrow$ $x_1 = \frac{1}{2}\left(m - \sqrt{m^2 + 4b}\right)$. Similarly, $x_2 = \frac{1}{2}\left(m + \sqrt{m^2 + 4b}\right)$. The area is then $\frac{1}{8}(x_2 - x_1)^3 = \frac{1}{8}\left(\sqrt{m^2 + 4b}\right)^3$, and is attained at the point $P\left(x_P, x_P^2\right) = P\left(\frac{1}{2}m, \frac{1}{4}m^2\right)$.

Note: Another way to get an expression for $f(x)$ is to use the formula for an area of a triangle in terms of the coordinates of the vertices: $f(x) = \frac{1}{2}\left[\left(x_2x_1^2 - x_1x_2^2\right) + \left(x_1x^2 - xx_1^2\right) + \left(xx_2^2 - x_2x^2\right)\right]$.

11. $f(x) = \left(a^2 + a - 6\right)\cos 2x + (a - 2)x + \cos 1 \quad\Rightarrow\quad f'(x) = -\left(a^2 + a - 6\right)\sin 2x\,(2) + (a - 2)$.

The derivative exists for all x, so the only possible critical points will occur where $f'(x) = 0 \quad\Leftrightarrow$ $2(a - 2)(a + 3)\sin 2x = a - 2 \quad\Leftrightarrow\quad$ either $a = 2$ or $2(a + 3)\sin 2x = 1$, with the latter implying that $\sin 2x = \dfrac{1}{2(a + 3)}$. Since the range of $\sin 2x$ is $[-1, 1]$, this equation has no solution whenever either $\dfrac{1}{2(a + 3)} < -1$ or $\dfrac{1}{2(a + 3)} > 1$. Solving these inequalities, we get $-\frac{7}{2} < a < -\frac{5}{2}$.

13. (a) Let $y = |AD|$, $x = |AB|$, and $1/x = |AC|$, so that $|AB| \cdot |AC| = 1$.

We compute the area $\mathcal{A}$ of $\triangle ABC$ in two ways. First,

$\mathcal{A} = \frac{1}{2}\,|AB|\,|AC|\sin\frac{2\pi}{3} = \frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$. Second,

$$\begin{aligned} \mathcal{A} &= (\text{area of } \triangle ABD) + (\text{area of } \triangle ACD) \\ &= \tfrac{1}{2}\,|AB|\,|AD|\sin\tfrac{\pi}{3} + \tfrac{1}{2}\,|AD|\,|AC|\sin\tfrac{\pi}{3} \\ &= \tfrac{1}{2}xy\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}y(1/x)\,\tfrac{\sqrt{3}}{2} = \tfrac{\sqrt{3}}{4}y(x + 1/x) \end{aligned}$$

Equating the two expressions for the area, we get $\frac{\sqrt{3}}{4}y\left(x + \dfrac{1}{x}\right) = \frac{\sqrt{3}}{4} \quad\Leftrightarrow\quad y = \dfrac{1}{x + 1/x} = \dfrac{x}{x^2 + 1}$, $x > 0$.

Another method: Use the Law of Sines on the triangles ABD and ABC. In $\triangle ABD$, we have $\angle A + \angle B + \angle D = 180^\circ \quad\Leftrightarrow\quad 60^\circ + \alpha + \angle D = 180^\circ \quad\Leftrightarrow\quad \angle D = 120^\circ - \alpha$. Thus,

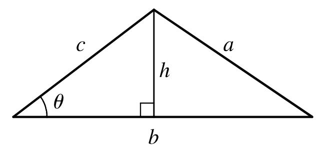
$$\frac{x}{y} = \frac{\sin(120^\circ - \alpha)}{\sin\alpha} = \frac{\sin 120^\circ \cos\alpha - \cos 120^\circ \sin\alpha}{\sin\alpha} = \frac{\frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha}{\sin\alpha} \quad\Rightarrow\quad \frac{x}{y} = \frac{\sqrt{3}}{2}\cot\alpha + \frac{1}{2},$$

and by a similar argument with $\triangle ABC$, $\frac{\sqrt{3}}{2}\cot\alpha = x^2 + \frac{1}{2}$. Eliminating $\cot\alpha$ gives $\dfrac{x}{y} = \left(x^2 + \frac{1}{2}\right) + \frac{1}{2} \quad\Rightarrow$ $y = \dfrac{x}{x^2 + 1}$, $x > 0$.

(b) We differentiate our expression for y with respect to x to find the maximum:

$\dfrac{dy}{dx} = \dfrac{(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} = 0$ when $x = 1$. This indicates a maximum by the First Derivative Test, since $y'(x) > 0$ for $0 < x < 1$ and $y'(x) < 0$ for $x > 1$, so the maximum value of y is $y(1) = \frac{1}{2}$.

15. (a) $A = \frac{1}{2}bh$ with $\sin\theta = h/c$, so $A = \frac{1}{2}bc\sin\theta$. But A is a constant, so differentiating this equation with respect to t, we get



$$\frac{dA}{dt} = 0 = \frac{1}{2}\left[bc\cos\theta\,\frac{d\theta}{dt} + b\,\frac{dc}{dt}\sin\theta + \frac{db}{dt}\,c\sin\theta\right] \quad\Rightarrow$$

$$bc\cos\theta\,\frac{d\theta}{dt} = -\sin\theta\left[b\,\frac{dc}{dt} + c\,\frac{db}{dt}\right] \quad\Rightarrow\quad \frac{d\theta}{dt} = -\tan\theta\left[\frac{1}{c}\frac{dc}{dt} + \frac{1}{b}\frac{db}{dt}\right].$$

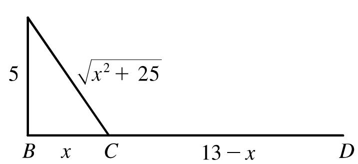
(b) We use the Law of Cosines to get the length of side a in terms of those of b and c, and then we differentiate implicitly with respect to t: $a^2 = b^2 + c^2 - 2bc\cos\theta \quad\Rightarrow$

$$2a\,\frac{da}{dt} = 2b\,\frac{db}{dt} + 2c\,\frac{dc}{dt} - 2\left[bc(-\sin\theta)\,\frac{d\theta}{dt} + b\,\frac{dc}{dt}\cos\theta + \frac{db}{dt}\,c\cos\theta\right] \quad\Rightarrow$$

$\dfrac{da}{dt} = \dfrac{1}{a}\left(b\,\dfrac{db}{dt} + c\,\dfrac{dc}{dt} + bc\sin\theta\,\dfrac{d\theta}{dt} - b\,\dfrac{dc}{dt}\cos\theta - c\,\dfrac{db}{dt}\cos\theta\right)$. Now we substitute our value of a from the Law of Cosines and the value of $d\theta/dt$ from part (a), and simplify (primes signify differentiation by t):

$$\begin{aligned}\frac{da}{dt} &= \frac{bb' + cc' + bc\sin\theta\,[-\tan\theta(c'/c + b'/b)] - (bc' + cb')(\cos\theta)}{\sqrt{b^2 + c^2 - 2bc\,\cos\theta}} \\ &= \frac{bb' + cc' - [\sin^2\theta(bc' + cb') + \cos^2\theta(bc' + cb')]/\cos\theta}{\sqrt{b^2 + c^2 - 2bc\cos\theta}} = \frac{bb' + cc' - (bc' + cb')\sec\theta}{\sqrt{b^2 + c^2 - 2bc\cos\theta}}\end{aligned}$$

17. (a)



If k = energy/km over land, then energy/km over water $= 1.4k$. So the total energy is $E = 1.4k\sqrt{25 + x^2} + k(13 - x)$, $0 \le x \le 13$, and so $\dfrac{dE}{dx} = \dfrac{1.4kx}{(25 + x^2)^{1/2}} - k$.

Set $\dfrac{dE}{dx} = 0$: $1.4kx = k(25 + x^2)^{1/2} \quad\Rightarrow\quad 1.96x^2 = x^2 + 25 \quad\Rightarrow\quad 0.96x^2 = 25 \quad\Rightarrow\quad x = \frac{5}{\sqrt{0.96}} \approx 5.1.$

Testing against the value of E at the endpoints: $E(0) = 1.4k(5) + 13k = 20k$, $E(5.1) \approx 17.9k$, $E(13) \approx 19.5k$. Thus, to minimize energy, the bird should fly to a point about 5.1 km from B.

(b) If W/L is large, the bird would fly to a point C that is closer to B than to D to minimize the energy used flying over water. If W/L is small, the bird would fly to a point C that is closer to D than to B to minimize the distance of the flight. $E = W\sqrt{25 + x^2} + L(13 - x) \quad\Rightarrow\quad \dfrac{dE}{dx} = \dfrac{Wx}{\sqrt{25 + x^2}} - L = 0$ when $\dfrac{W}{L} = \dfrac{\sqrt{25 + x^2}}{x}$. By the same sort of argument as in part (a), this ratio will give the minimal expenditure of energy if the bird heads for the point x km from B.

(c) For flight direct to D, $x = 13$, so from part (b), $W/L = \frac{\sqrt{25+13^2}}{13} \approx 1.07$. There is no value of W/L for which the bird should fly directly to B. But note that $\lim\limits_{x\to 0^+} (W/L) = \infty$, so if the point at which E is a minimum is close to B, then W/L is large.

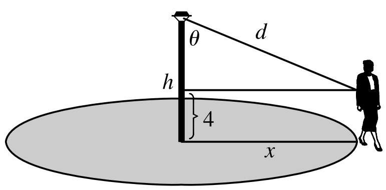
(d) Assuming that the birds instinctively choose the path that minimizes the energy expenditure, we can use the equation for $dE/dx = 0$ from part (a) with $1.4k = c$, $x = 4$, and $k = 1$: $c(4) = 1 \cdot \left(25 + 4^2\right)^{1/2} \quad\Rightarrow$ $c = \sqrt{41}/4 \approx 1.6$.

19. (a) $I = \dfrac{k\cos\theta}{d^2} = \dfrac{k(h/d)}{d^2} = k\dfrac{h}{d^3} = k\dfrac{h}{\left(\sqrt{40^2+h^2}\right)^3} = k\dfrac{h}{(1600+h^2)^{3/2}} \quad\Rightarrow$

$$\begin{aligned}\frac{dI}{dh} &= k\,\frac{(1600+h^2)^{3/2} - h\frac{3}{2}(1600+h^2)^{1/2}\cdot 2h}{[(1600+h^2)^{3/2}]^2} = \frac{k(1600+h^2)^{1/2}(1600+h^2-3h^2)}{(1600+h^2)^{3/2}}\\ &= \frac{k(1600-2h^2)}{(1600+h^2)^{5/2}} \qquad [k \text{ is the constant of proportionality}]\end{aligned}$$

Set $dI/dh = 0$: $1600 - 2h^2 = 0 \quad\Rightarrow\quad h^2 = 800 \quad\Rightarrow\quad h = \sqrt{800} = 20\sqrt{2}$. By the First Derivative Test, I has a local maximum at $h = 20\sqrt{2} \approx 28$ ft.

(b)



$$\frac{dx}{dt} = 4 \text{ ft/s}$$

$$\begin{aligned}I &= \frac{k\cos\theta}{d^2} = \frac{k[(h-4)/d]}{d^2} = \frac{k(h-4)}{d^3}\\ &= \frac{k(h-4)}{[(h-4)^2+x^2]^{3/2}}\\ &= k(h-4)\left[(h-4)^2+x^2\right]^{-3/2}\end{aligned}$$

$$\begin{aligned}\frac{dI}{dt} &= \frac{dI}{dx}\cdot\frac{dx}{dt} = k(h-4)\left(-\tfrac{3}{2}\right)\left[(h-4)^2+x^2\right]^{-5/2}\cdot 2x\cdot\frac{dx}{dt}\\ &= k(h-4)(-3x)\left[(h-4)^2+x^2\right]^{-5/2}\cdot 4 = \frac{-12xk(h-4)}{[(h-4)^2+x^2]^{5/2}}\end{aligned}$$

$$\left.\frac{dI}{dt}\right|_{x=40} = -\frac{480k(h-4)}{[(h-4)^2+1600]^{5/2}}$$

21. (a) Distance $=$ rate $\times$ time, so time $=$ distance/rate. $T_1 = \dfrac{D}{c_1}$,

$$T_2 = \frac{2\,|PR|}{c_1} + \frac{|RS|}{c_2} = \frac{2h\sec\theta}{c_1} + \frac{D - 2h\tan\theta}{c_2},\ T_3 = \frac{2\sqrt{h^2+D^2/4}}{c_1} = \frac{\sqrt{4h^2+D^2}}{c_1}.$$

(b) $\dfrac{dT_2}{d\theta} = \dfrac{2h}{c_1}\cdot\sec\theta\,\tan\theta - \dfrac{2h}{c_2}\sec^2\theta = 0$ when $2h\sec\theta\left(\dfrac{1}{c_1}\tan\theta - \dfrac{1}{c_2}\sec\theta\right) = 0 \quad\Rightarrow$

$\dfrac{1}{c_1}\dfrac{\sin\theta}{\cos\theta} - \dfrac{1}{c_2}\dfrac{1}{\cos\theta} = 0 \quad\Rightarrow\quad \dfrac{\sin\theta}{c_1\cos\theta} = \dfrac{1}{c_2\cos\theta} \quad\Rightarrow\quad \sin\theta = \dfrac{c_1}{c_2}$. The First Derivative Test shows that this gives a minimum.

(c) Using part (a) with $D = 1$ and $T_1 = 0.26$, we have $T_1 = \dfrac{D}{c_1} \quad\Rightarrow\quad c_1 = \frac{1}{0.26} \approx 3.85$ km/s. $T_3 = \dfrac{\sqrt{4h^2+D^2}}{c_1}$ $\Rightarrow\quad 4h^2 + D^2 = T_3^2c_1^2 \quad\Rightarrow\quad h = \frac{1}{2}\sqrt{T_3^2c_1^2 - D^2} = \frac{1}{2}\sqrt{(0.34)^2(1/0.26)^2 - 1^2} \approx 0.42$ km. To find c_2, we use $\sin\theta = \dfrac{c_1}{c_2}$ from part (b) and $T_2 = \dfrac{2h\sec\theta}{c_1} + \dfrac{D - 2h\tan\theta}{c_2}$ from part (a). From the figure,

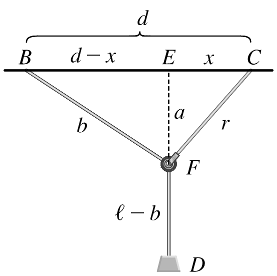
$\sin\theta = \dfrac{c_1}{c_2} \quad\Rightarrow\quad \sec\theta = \dfrac{c_2}{\sqrt{c_2^2 - c_1^2}}$ and $\tan\theta = \dfrac{c_1}{\sqrt{c_2^2-c_1^2}}$, so

$T_2 = \dfrac{2hc_2}{c_1\sqrt{c_2^2-c_1^2}} + \dfrac{D\sqrt{c_2^2-c_1^2} - 2hc_1}{c_2\sqrt{c_2^2-c_1^2}}$.

Using the values for T_2 [given as 0.32], h, c_1, and D, we can graph

$\mathrm{Y}_1 = T_2$ and $\mathrm{Y}_2 = \dfrac{2hc_2}{c_1\sqrt{c_2^2-c_1^2}} + \dfrac{D\sqrt{c_2^2-c_1^2} - 2hc_1}{c_2\sqrt{c_2^2-c_1^2}}$ and find their intersection points. Doing so gives us

$c_2 \approx 4.10$ and 7.66, but if $c_2 = 4.10$, then $\theta = \arcsin(c_1/c_2) \approx 69.6^\circ$, which implies that point S is to the left of point R in the diagram. So $c_2 = 7.66$ km/s.

23. 

Let $a = |EF|$ and $b = |BF|$ as shown in the figure.
Since $\ell = |BF| + |FD|$, $|FD| = \ell - b$. Now

$$\begin{aligned}|ED| &= |EF| + |FD| = a + \ell - b \\ &= \sqrt{r^2-x^2} + \ell - \sqrt{(d-x)^2 + a^2} \\ &= \sqrt{r^2-x^2} + \ell - \sqrt{(d-x)^2 + \left(\sqrt{r^2-x^2}\right)^2} \\ &= \sqrt{r^2-x^2} + \ell - \sqrt{d^2 - 2dx + x^2 + r^2 - x^2}\end{aligned}$$

Let $f(x) = \sqrt{r^2-x^2} + \ell - \sqrt{d^2 + r^2 - 2dx}$.

$f'(x) = \frac{1}{2}\left(r^2-x^2\right)^{-1/2}(-2x) - \frac{1}{2}\left(d^2+r^2-2dx\right)^{-1/2}(-2d) = \dfrac{-x}{\sqrt{r^2-x^2}} + \dfrac{d}{\sqrt{d^2+r^2-2dx}}$.

$f'(x) = 0 \quad\Rightarrow\quad \dfrac{x}{\sqrt{r^2-x^2}} = \dfrac{d}{\sqrt{d^2+r^2-2dx}} \quad\Rightarrow\quad \dfrac{x^2}{r^2-x^2} = \dfrac{d^2}{d^2+r^2-2dx} \quad\Rightarrow$

$d^2x^2 + r^2x^2 - 2dx^3 = d^2r^2 - d^2x^2 \quad\Rightarrow\quad 0 = 2dx^3 - 2d^2x^2 - r^2x^2 + d^2r^2 \quad\Rightarrow$

$0 = 2dx^2(x-d) - r^2\left(x^2-d^2\right) \quad\Rightarrow\quad 0 = 2dx^2(x-d) - r^2(x+d)(x-d) \quad\Rightarrow$

$0 = (x-d)\left[2dx^2 - r^2(x+d)\right]$

But $d > r > x$, so $x \neq d$. Thus, we solve $2dx^2 - r^2x - dr^2 = 0$ for x:

$x = \dfrac{-\left(-r^2\right) \pm \sqrt{\left(-r^2\right)^2 - 4(2d)\left(-dr^2\right)}}{2(2d)} = \dfrac{r^2 \pm \sqrt{r^4 + 8d^2r^2}}{4d}$. Because $\sqrt{r^4+8d^2r^2} > r^2$, the "negative" can be discarded. Thus,

$$\begin{aligned}x &= \frac{r^2 + \sqrt{r^2}\,\sqrt{r^2+8d^2}}{4d} = \frac{r^2 + r\sqrt{r^2+8d^2}}{4d} \qquad (r>0) \\ &= \frac{r}{4d}\left(r + \sqrt{r^2+8d^2}\right)\end{aligned}$$

The maximum value of $|ED|$ occurs at this value of x.

25. $V = \frac{4}{3}\pi r^3 \quad\Leftrightarrow\quad \dfrac{dV}{dt} = 4\pi r^2\,\dfrac{dr}{dt}$. But $\dfrac{dV}{dt}$ is proportional to the surface area, so $\dfrac{dV}{dt} = k\cdot 4\pi r^2$ for some constant k. Therefore, $4\pi r^2\,\dfrac{dr}{dt} = k\cdot 4\pi r^2 \quad\Leftrightarrow\quad \dfrac{dr}{dt} = k =$ constant. An antiderivative of k with respect to t

is kt, so $r = kt + C$. When $t = 0$, the radius r must equal the original radius r_0, so $C = r_0$, and $r = kt + r_0$. To find k we use the fact that when $t = 3$, $r = 3k + r_0$ and $V = \frac{1}{2}V_0 \quad \Rightarrow \quad \frac{4}{3}\pi(3k + r_0)^3 = \frac{1}{2} \cdot \frac{4}{3}\pi r_0^3 \quad \Rightarrow$ $(3k + r_0)^3 = \frac{1}{2}r_0^3 \quad \Rightarrow \quad 3k + r_0 = \frac{1}{\sqrt[3]{2}}r_0 \quad \Rightarrow \quad k = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)$. Since $r = kt + r_0$, $r = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0$. When the snowball has melted completely we have $r = 0 \quad \Rightarrow$ $\frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0 = 0$ which gives $t = \dfrac{3\sqrt[3]{2}}{\sqrt[3]{2} - 1}$. Hence, it takes $\dfrac{3\sqrt[3]{2}}{\sqrt[3]{2} - 1} - 3 = \dfrac{3}{\sqrt[3]{2} - 1} \approx 11$ h 33 min longer.