CHALLENGE PROBLEMS

CHAPTER 7

A Click here for answers. S Click here for solutions.

1. (a) Find a positive continuous function f such that the area under the graph of f from 0 to t is $A(t) = t^3$ for all $t > 0$.
(b) A solid is generated by rotating about the x-axis the region under the curve $y = f(x)$, where f is a positive function and $x \geqslant 0$. The volume generated by the part of the curve from $x = 0$ to $x = b$ is b^2 for all $b > 0$. Find the function f.

2. There is a line through the origin that divides the region bounded by the parabola $y = x - x^2$ and the x-axis into two regions with equal area. What is the slope of that line?

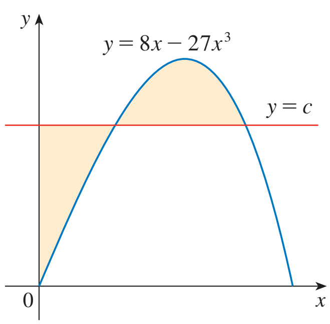
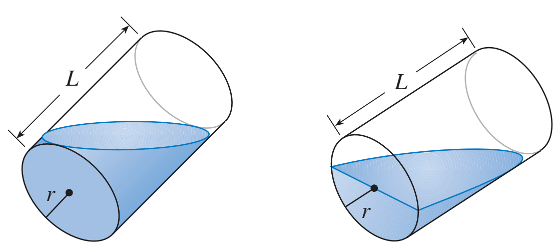


FIGURE FOR PROBLEM 3

3. The figure shows a horizontal line $y = c$ intersecting the curve $y = 8x - 27x^3$. Find the number c such that the areas of the shaded regions are equal.

4. A cylindrical glass of radius r and height L is filled with water and then tilted until the water remaining in the glass exactly covers its base.
(a) Determine a way to "slice" the water into parallel rectangular cross-sections and then *set up* a definite integral for the volume of the water in the glass.
(b) Determine a way to "slice" the water into parallel cross-sections that are trapezoids and then *set up* a definite integral for the volume of the water.
(c) Find the volume of water in the glass by evaluating one of the integrals in part (a) or part (b).
(d) Find the volume of the water in the glass from purely geometric considerations.
(e) Suppose the glass is tilted until the water exactly covers half the base. In what direction can you "slice" the water into triangular cross-sections? Rectangular cross-sections? Cross-sections that are segments of circles? Find the volume of water in the glass.



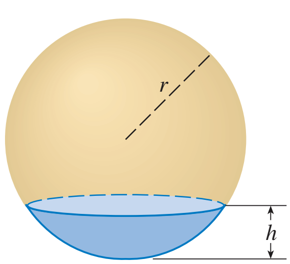


FIGURE FOR PROBLEM 5

5. (a) Show that the volume of a segment of height h of a sphere of radius r is

$$V = \tfrac{1}{3}\pi h^2(3r - h)$$

(b) Show that if a sphere of radius 1 is sliced by a plane at a distance x from the center in such a way that the volume of one segment is twice the volume of the other, then x is a solution of the equation

$$3x^3 - 9x + 2 = 0$$

where $0 < x < 1$. Use Newton's method to find x accurate to four decimal places.
(c) Using the formula for the volume of a segment of a sphere, it can be shown that the depth x to which a floating sphere of radius r sinks in water is a root of the equation

$$x^3 - 3rx^2 + 4r^3s = 0$$

where s is the specific gravity of the sphere. Suppose a wooden sphere of radius 0.5 m has specific gravity 0.75. Calculate, to four-decimal-place accuracy, the depth to which the sphere will sink.
(d) A hemispherical bowl has radius 5 inches and water is running into the bowl at the rate of 0.2 in^3/s.
 (i) How fast is the water level in the bowl rising at the instant the water is 3 inches deep?
 (ii) At a certain instant, the water is 4 inches deep. How long will it take to fill the bowl?

6. Archimedes' Principle states that the buoyant force on an object partially or fully submerged in a fluid is equal to the weight of the fluid that the object displaces. Thus, for an object of density ρ_0 floating partly submerged in a fluid of density ρ_f, the buoyant force is given by $F = \rho_f g \int_{-h}^{0} A(y)\,dy$, where g is the acceleration due to gravity and $A(y)$ is the area of a typical cross-section of the object. The weight of the object is given by

$$W = \rho_0 g \int_{-h}^{L-h} A(y)\,dy$$

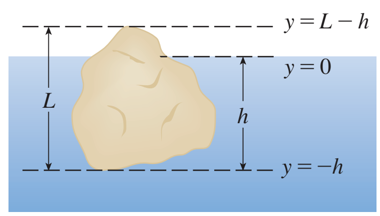
(a) Show that the percentage of the volume of the object above the surface of the liquid is

$$100\,\frac{\rho_f - \rho_0}{\rho_f}$$

(b) The density of ice is 917 kg/m^3 and the density of seawater is 1030 kg/m^3. What percentage of the volume of an iceberg is above water?

(c) An ice cube floats in a glass filled to the brim with water. Does the water overflow when the ice melts?

(d) A sphere of radius 0.4 m and having negligible weight is floating in a large freshwater lake. How much work is required to completely submerge the sphere? The density of the water is 1000 kg/m^3.



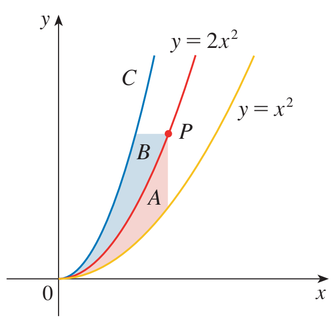


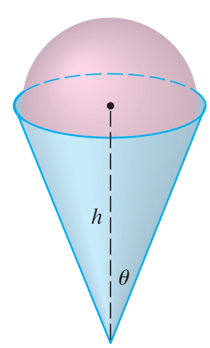
FIGURE FOR PROBLEM 9

7. Water in an open bowl evaporates at a rate proportional to the area of the surface of the water. (This means that the rate of decrease of the volume is proportional to the area of the surface.) Show that the depth of the water decreases at a constant rate, regardless of the shape of the bowl.

8. A sphere of radius 1 overlaps a smaller sphere of radius r in such a way that their intersection is a circle of radius r. (In other words, they intersect in a great circle of the small sphere.) Find r so that the volume inside the small sphere and outside the large sphere is as large as possible.

9. The figure shows a curve C with the property that, for every point P on the middle curve $y = 2x^2$, the areas A and B are equal. Find an equation for C.

10. A paper drinking cup filled with water has the shape of a cone with height h and semivertical angle θ (see the figure). A ball is placed carefully in the cup, thereby displacing some of the water and making it overflow. What is the radius of the ball that causes the greatest volume of water to spill out of the cup?



11. Find the area of the region $S = \{(x, y) \mid x \geq 0,\ y \leq 1,\ x^2 + y^2 \leq 4y\}$.

12. Find the centroid of the region enclosed by the loop of the curve $y^2 = x^3 - x^4$.

13. Suppose that the density of seawater, $\rho = \rho(z)$, varies with the depth z below the surface.

(a) Show that the hydrostatic pressure is governed by the differential equation

$$\frac{dP}{dz} = \rho(z)g$$

where g is the acceleration due to gravity. Let P_0 and ρ_0 be the pressure and density at $z = 0$. Express the pressure at depth z as an integral.

(b) Suppose the density of seawater at depth z is given by $\rho = \rho_0 e^{z/H}$, where H is a positive constant. Find the total force, expressed as an integral, exerted on a vertical circular porthole of radius r whose center is located at a distance $L > r$ below the surface.

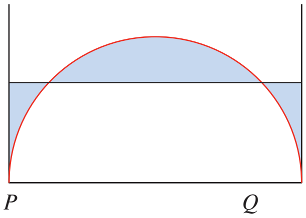


FIGURE FOR PROBLEM 14

14. The figure shows a semicircle with radius 1, horizontal diameter PQ, and tangent lines at P and Q. At what height above the diameter should the horizontal line be placed so as to minimize the shaded area?

15. Let P be a pyramid with a square base of side $2b$ and suppose that S is a sphere with its center on the base of P and is tangent to all eight edges of P. Find the height of P. Then find the volume of the intersection of S and P.

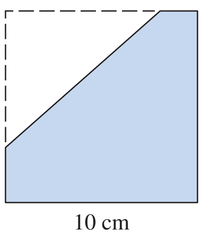


FIGURE FOR PROBLEM 18

16. Consider a flat metal plate to be placed vertically under water with its top 2 m below the surface of the water. Determine a shape for the plate so that if the plate is divided into any number of horizontal strips of equal height, the hydrostatic force on each strip is the same.

17. A uniform disk with radius 1 m is to be cut by a line so that the center of mass of the smaller piece lies halfway along a radius. How close to the center of the disk should the cut be made? (Express your answer correct to two decimal places.)

18. A triangle with area 30 cm^2 is cut from a corner of a square with side 10 cm, as shown in the figure. If the centroid of the remaining region is 4 cm from the right side of the square, how far is it from the bottom of the square?

19. Find all functions f such that f' is continuous and

$$[f(x)]^2 = 100 + \int_0^x \{[f(t)]^2 + [f'(t)]^2\}\,dt \qquad \text{for all real } x$$

20. A student forgot the Product Rule for differentiation and made the mistake of thinking that $(fg)' = f'g'$. However, he was lucky and got the correct answer. The function f that he used was $f(x) = e^{x^2}$ and the domain of his problem was the interval $(\frac{1}{2}, \infty)$. What was the function g?

21. Let f be a function with the property that $f(0) = 1$, $f'(0) = 1$, and $f(a + b) = f(a)f(b)$ for all real numbers a and b. Show that $f'(x) = f(x)$ for all x and deduce that $f(x) = e^x$.

22. Snow began to fall during the morning of February 2 and continued steadily into the afternoon. At noon a snowplow began removing snow from a road at a constant rate. The plow traveled 6 km from noon to 1 P.M. but only 3 km from 1 P.M. to 2 P.M. When did the snow begin to fall? [*Hints:* To get started, let t be the time measured in hours after noon; let $x(t)$ be the distance traveled by the plow at time t; then the speed of the plow is dx/dt. Let b be the number of hours before noon that it began to snow. Find an expression for the height of the snow at time t. Then use the given information that the rate of removal R (in m^3/h) is constant.]

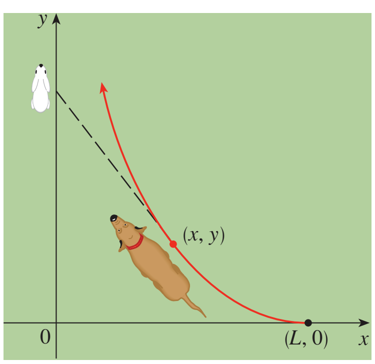


FIGURE FOR PROBLEM 23

23. A dog sees a rabbit running in a straight line across an open field and gives chase. In a rectangular coordinate system (as shown in the figure), assume:

(i) The rabbit is at the origin and the dog is at the point $(L, 0)$ at the instant the dog first sees the rabbit.

(ii) The rabbit runs up the y-axis and the dog always runs straight for the rabbit.

(iii) The dog runs at the same speed as the rabbit.

(a) Show that the dog's path is the graph of the function $y = f(x)$, where y satisfies the differential equation

$$x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

(b) Determine the solution of the equation in part (a) that satisfies the initial conditions $y = y' = 0$ when $x = L$. [*Hint:* Let $z = dy/dx$ in the differential equation and solve the resulting first-order equation to find z; then integrate z to find y.]

(c) Does the dog ever catch the rabbit?

24. (a) Suppose that the dog in Problem 23 runs twice as fast as the rabbit. Find a differential equation for the path of the dog. Then solve it to find the point where the dog catches the rabbit.

(b) Suppose the dog runs half as fast as the rabbit. How close does the dog get to the rabbit? What are their positions when they are closest?

25. A planning engineer for a new alum plant must present some estimates to his company regarding the capacity of a silo designed to contain bauxite ore until it is processed into alum. The ore resembles pink talcum powder and is poured from a conveyor at the top of the silo. The silo is a cylinder 100 ft high with a radius of 200 ft. The conveyor carries $60{,}000\pi\ \mathrm{ft}^3/\mathrm{h}$ and the ore maintains a conical shape whose radius is 1.5 times its height.

(a) If, at a certain time t, the pile is 60 ft high, how long will it take for the pile to reach the top of the silo?

(b) Management wants to know how much room will be left in the floor area of the silo when the pile is 60 ft high. How fast is the floor area of the pile growing at that height?

(c) Suppose a loader starts removing the ore at the rate of $20{,}000\pi\ \mathrm{ft}^3/\mathrm{h}$ when the height of the pile reaches 90 ft. Suppose, also, that the pile continues to maintain its shape. How long will it take for the pile to reach the top of the silo under these conditions?

26. Find the curve that passes through the point $(3, 2)$ and has the property that if the tangent line is drawn at any point P on the curve, then the part of the tangent line that lies in the first quadrant is bisected at P.

27. Recall that the normal line to a curve at a point P on the curve is the line that passes through P and is perpendicular to the tangent line at P. Find the curve that passes through the point $(3, 2)$ and has the property that if the normal line is drawn at any point on the curve, then the y-intercept of the normal line is always 6.

28. Find all curves with the property that if the normal line is drawn at any point P on the curve, then the part of the normal line between P and the x-axis is bisected by the y-axis.

ANSWERS

S Solutions

1. (a) $f(t) = 3t^2$ (b) $f(x) = \sqrt{2x/\pi}$ **3.** $\frac{32}{27}$

5. (b) 0.2261 (c) 0.6736 m (d) (i) $1/(105\pi) \approx 0.003$ in/s (ii) $370\pi/3$ s ≈ 6.5 min

9. $y = \frac{32}{9}x^2$ **11.** $2\pi/3 - \sqrt{3}/2$

13. (a) $P(z) = P_0 + g\int_0^z \rho(x)\,dx$ (b) $(P_0 - \rho_0 gH)(\pi r^2) + \rho_0 gHe^{L/H}\int_{-r}^{r} e^{x/H} \cdot 2\sqrt{r^2 - x^2}\,dx$

15. Height $\sqrt{2}\,b$, volume $\left(\frac{28}{27}\sqrt{6} - 2\right)\pi b^3$ **17.** 0.14 m **19.** $f(x) = \pm 10e^x$

23. (b) $f(x) = (x^2 - L^2)/(4L) - (L/2)\ln(x/L)$ (c) No

25. (a) 9.8 h (b) $31{,}900\pi \approx 100{,}000$ ft^2; 6283 ft^2/h (c) 5.1 h **27.** $x^2 + (y - 6)^2 = 25$

SOLUTIONS

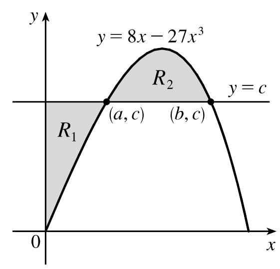
E Exercises

1. (a) The area under the graph of f from 0 to t is equal to $\int_0^t f(x)\,dx$, so the requirement is that $\int_0^t f(x)\,dx = t^3$ for all t. We differentiate both sides of this equation with respect to t (with the help of FTC1) to get $f(t) = 3t^2$. This function is positive and continuous, as required.

(b) The volume generated from $x = 0$ to $x = b$ is $\int_0^b \pi[f(x)]^2\,dx$. Hence, we are given that $b^2 = \int_0^b \pi[f(x)]^2\,dx$ for all $b > 0$. Differentiating both sides of this equation with respect to b using the Fundamental Theorem of Calculus gives

$2b = \pi[f(b)]^2 \quad\Rightarrow\quad f(b) = \sqrt{2b/\pi}$, since f is positive. Therefore, $f(x) = \sqrt{2x/\pi}$.

3. Let a and b be the x-coordinates of the points where the line intersects the curve. From the figure, $R_1 = R_2 \quad\Rightarrow$



$$\int_0^a \left[c - \left(8x - 27x^3\right)\right] dx = \int_a^b \left[\left(8x - 27x^3\right) - c\right] dx$$

$$\left[cx - 4x^2 + \tfrac{27}{4}x^4\right]_0^a = \left[4x^2 - \tfrac{27}{4}x^4 - cx\right]_a^b$$

$$ac - 4a^2 + \tfrac{27}{4}a^4 = \left(4b^2 - \tfrac{27}{4}b^4 - bc\right) - \left(4a^2 - \tfrac{27}{4}a^4 - ac\right)$$

$$0 = 4b^2 - \tfrac{27}{4}b^4 - bc = 4b^2 - \tfrac{27}{4}b^4 - b\left(8b - 27b^3\right)$$

$$= 4b^2 - \tfrac{27}{4}b^4 - 8b^2 + 27b^4 = \tfrac{81}{4}b^4 - 4b^2$$

$$= b^2\left(\tfrac{81}{4}b^2 - 4\right)$$

So for $b > 0$, $b^2 = \frac{16}{81} \quad\Rightarrow\quad b = \frac{4}{9}$. Thus, $c = 8b - 27b^3 = 8\left(\frac{4}{9}\right) - 27\left(\frac{64}{729}\right) = \frac{32}{9} - \frac{64}{27} = \frac{32}{27}$.

5. (a) $V = \pi h^2(r - h/3) = \frac{1}{3}\pi h^2(3r - h)$. See the solution to Exercise 7.2.27.

(b) The smaller segment has height $h = 1 - x$ and so by part (a) its volume is $V = \frac{1}{3}\pi(1-x)^2\,[3(1) - (1-x)] = \frac{1}{3}\pi(x-1)^2(x+2)$. This volume must be $\frac{1}{3}$ of the total volume of the sphere, which is $\frac{4}{3}\pi(1)^3$. So $\frac{1}{3}\pi(x-1)^2(x+2) = \frac{1}{3}\left(\frac{4}{3}\pi\right) \quad\Rightarrow\quad \left(x^2 - 2x + 1\right)(x+2) = \frac{4}{3} \quad\Rightarrow$ $x^3 - 3x + 2 = \frac{4}{3} \quad\Rightarrow\quad 3x^3 - 9x + 2 = 0$. Using Newton's method with $f(x) = 3x^3 - 9x + 2$, $f'(x) = 9x^2 - 9$, we get $x_{n+1} = x_n - \dfrac{3x_n^3 - 9x_n + 2}{9x_n^2 - 9}$. Taking $x_1 = 0$, we get $x_2 \approx 0.2222$, and $x_3 \approx 0.2261 \approx x_4$, so, correct to four decimal places, $x \approx 0.2261$.

(c) With $r = 0.5$ and $s = 0.75$, the equation $x^3 - 3rx^2 + 4r^3 s = 0$ becomes $x^3 - 3(0.5)x^2 + 4(0.5)^3(0.75) = 0$ $\Rightarrow\quad x^3 - \frac{3}{2}x^2 + 4\left(\frac{1}{8}\right)\frac{3}{4} = 0 \quad\Rightarrow\quad 8x^3 - 12x^2 + 3 = 0$. We use Newton's method with $f(x) = 8x^3 - 12x^2 + 3$, $f'(x) = 24x^2 - 24x$, so $x_{n+1} = x_n - \dfrac{8x_n^3 - 12x_n^2 + 3}{24x_n^2 - 24x_n}$. Take $x_1 = 0.5$. Then $x_2 \approx 0.6667$, and $x_3 \approx 0.6736 \approx x_4$. So to four decimal places the depth is 0.6736 m.

(d) (i) From part (a) with $r = 5$ in., the volume of water in the bowl is

$V = \frac{1}{3}\pi h^2(3r - h) = \frac{1}{3}\pi h^2(15 - h) = 5\pi h^2 - \frac{1}{3}\pi h^3$. We are given that $\dfrac{dV}{dt} = 0.2\ \text{m}^3/\text{s}$ and we want to find $\dfrac{dh}{dt}$ when $h = 3$. Now $\dfrac{dV}{dt} = 10\pi h\,\dfrac{dh}{dt} - \pi h^2\dfrac{dh}{dt}$, so $\dfrac{dh}{dt} = \dfrac{0.2}{\pi(10h - h^2)}$. When $h = 3$, we have

$\dfrac{dh}{dt} = \dfrac{0.2}{\pi(10 \cdot 3 - 3^2)} = \dfrac{1}{105\pi} \approx 0.003$ in/s.

(ii) From part (a), the volume of water required to fill the bowl from the instant that the water is 4 in. deep is $V = \frac{1}{2} \cdot \frac{4}{3}\pi(5)^3 - \frac{1}{3}\pi(4)^2(15 - 4) = \frac{2}{3} \cdot 125\pi - \frac{16}{3} \cdot 11\pi = \frac{74}{3}\pi$. To find the time required to fill the bowl we divide this volume by the rate: Time $= \frac{74\pi/3}{0.2} = \frac{370\pi}{3} \approx 387$ s ≈ 6.5 min

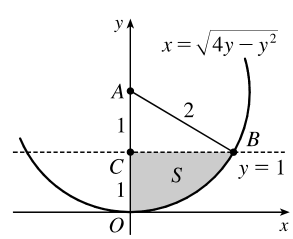
7. We are given that the rate of change of the volume of water is $\dfrac{dV}{dt} = -kA(x)$, where k is some positive constant and $A(x)$ is the area of the surface when the water has depth x. Now we are concerned with the rate of change of the depth of the water with respect to time, that is, $\dfrac{dx}{dt}$. But by the Chain Rule, $\dfrac{dV}{dt} = \dfrac{dV}{dx}\dfrac{dx}{dt}$, so the first equation can be written $\dfrac{dV}{dx}\dfrac{dx}{dt} = -kA(x)$ $(\star)$. Also, we know that the total volume of water up to a depth x is $V(x) = \int_0^x A(s)\,ds$, where $A(s)$ is the area of a cross-section of the water at a depth s. Differentiating this equation with respect to x, we get $dV/dx = A(x)$. Substituting this into equation $\star$, we get $A(x)(dx/dt) = -kA(x) \quad\Rightarrow$ $dx/dt = -k$, a constant.

9. We must find expressions for the areas A and B, and then set them equal and see what this says about the curve C. If $P = \left(a, 2a^2\right)$, then area A is just $\int_0^a \left(2x^2 - x^2\right) dx = \int_0^a x^2\,dx = \frac{1}{3}a^3$. To find area B, we use y as the variable of integration. So we find the equation of the middle curve as a function of y: $y = 2x^2 \quad\Leftrightarrow\quad x = \sqrt{y/2}$, since we are concerned with the first quadrant only. We can express area B as

$\int_0^{2a^2}\left[\sqrt{y/2} - C(y)\right] dy = \left[\frac{4}{3}(y/2)^{3/2}\right]_0^{2a^2} - \int_0^{2a^2} C(y)\,dy = \frac{4}{3}a^3 - \int_0^{2a^2} C(y)\,dy$, where $C(y)$ is the function with graph C. Setting $A = B$, we get $\frac{1}{3}a^3 = \frac{4}{3}a^3 - \int_0^{2a^2} C(y)\,dy \quad\Leftrightarrow\quad \int_0^{2a^2} C(y)\,dy = a^3$. Now we differentiate this equation with respect to a using the Chain Rule and the Fundamental Theorem:

$C(2a^2)(4a) = 3a^2 \quad\Rightarrow\quad C(y) = \frac{3}{4}\sqrt{y/2}$, where $y = 2a^2$. Now we can solve for y: $x = \frac{3}{4}\sqrt{y/2} \quad\Rightarrow$ $x^2 = \frac{9}{16}(y/2) \quad\Rightarrow\quad y = \frac{32}{9}x^2$.

11. $x^2 + y^2 \le 4y \quad\Leftrightarrow\quad x^2 + (y-2)^2 \le 4$, so S is part of a circle, as shown in the diagram. The area of S is



$$\begin{aligned}\int_0^1 \sqrt{4y - y^2}\,dy &\overset{113}{=} \left[\tfrac{y-2}{2}\sqrt{4y - y^2} + 2\cos^{-1}\left(\tfrac{2-y}{2}\right)\right]_0^1 \quad [a = 2]\\ &= -\tfrac{1}{2}\sqrt{3} + 2\cos^{-1}\left(\tfrac{1}{2}\right) - 2\cos^{-1} 1\\ &= -\tfrac{\sqrt{3}}{2} + 2\left(\tfrac{\pi}{3}\right) - 2(0) = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}\end{aligned}$$

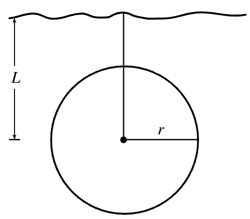
Another method (without calculus): Note that $\theta = \angle CAB = \frac{\pi}{3}$, so the area is

$$(\text{area of sector } OAB) - (\text{area of } \triangle ABC) = \tfrac{1}{2}\left(2^2\right)\tfrac{\pi}{3} - \tfrac{1}{2}(1)\sqrt{3} = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}$$

13. (a) Choose a vertical x-axis pointing downward with its origin at the surface. In order to calculate the pressure at depth z, consider n subintervals of the interval $[0, z]$ by points x_i and choose a point $x_i^* \in [x_{i-1}, x_i]$ for each i. The thin layer of water lying between depth x_{i-1} and depth x_i has a density of approximately $\rho(x_i^*)$, so the weight of a piece of that layer with unit cross-sectional area is $\rho(x_i^*)g\,\Delta x$. The total weight of a column of water extending from the surface to depth z (with unit cross-sectional area) would be approximately $\sum\limits_{i=1}^{n} \rho(x_i^*)g\,\Delta x$.

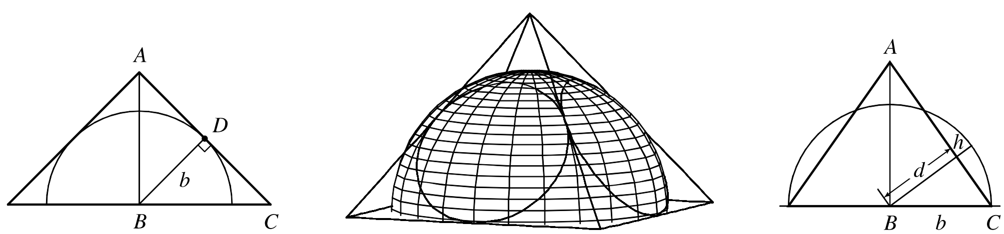
The estimate becomes exact if we take the limit as $n \to \infty$; weight (or force) per unit area at depth z is $W = \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*) g\, \Delta x$. In other words, $P(z) = \int_0^z \rho(x) g\, dx$. More generally, if we make no assumptions about the location of the origin, then $P(z) = P_0 + \int_0^z \rho(x) g\, dx$, where P_0 is the pressure at $x = 0$. Differentiating, we get $dP/dz = \rho(z) g$.

(b)



$$\begin{aligned} F &= \int_{-r}^{r} P(L+x) \cdot 2\sqrt{r^2 - x^2}\, dx \\ &= \int_{-r}^{r} \left(P_0 + \int_0^{L+x} \rho_0 e^{z/H} g\, dz\right) \cdot 2\sqrt{r^2 - x^2}\, dx \\ &= P_0 \int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx + \rho_0 g H \int_{-r}^{r} \left(e^{(L+x)/H} - 1\right) \cdot 2\sqrt{r^2 - x^2}\, dx \\ &= (P_0 - \rho_0 g H) \int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx + \rho_0 g H \int_{-r}^{r} e^{(L+x)/H} \cdot 2\sqrt{r^2 - x^2}\, dx \\ &= (P_0 - \rho_0 g H)\left(\pi r^2\right) + \rho_0 g H e^{L/H} \int_{-r}^{r} e^{x/H} \cdot 2\sqrt{r^2 - x^2}\, dx \end{aligned}$$

15. To find the height of the pyramid, we use similar triangles. The first figure shows a cross-section of the pyramid passing through the top and through two opposite corners of the square base. Now $|BD| = b$, since it is a radius of the sphere, which has diameter $2b$ since it is tangent to the opposite sides of the square base. Also, $|AD| = b$ since $\triangle ADB$ is isosceles. So the height is $|AB| = \sqrt{b^2 + b^2} = \sqrt{2}b$.

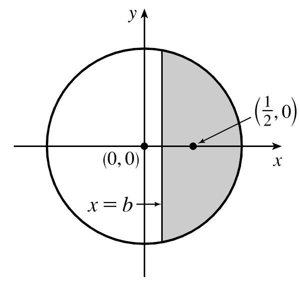


We first observe that the shared volume is equal to half the volume of the sphere, minus the sum of the four equal volumes (caps of the sphere) cut off by the triangular faces of the pyramid. See Exercise 7.2.27 for a derivation of the formula for the volume of a cap of a sphere. To use the formula, we need to find the perpendicular distance h of each triangular face from the surface of the sphere. We first find the distance d from the center of the sphere to one of the triangular faces. The third figure shows a cross-section of the pyramid through the top and through the midpoints of opposite sides of the square base. From similar triangles we find that

$$\frac{d}{b} = \frac{|AB|}{|AC|} = \frac{\sqrt{2}\, b}{\sqrt{b^2 + \left(\sqrt{2}\, b\right)^2}} \quad \Rightarrow \quad d = \frac{\sqrt{2}\, b^2}{\sqrt{3b^2}} = \frac{\sqrt{6}}{3}\, b$$

So $h = b - d = b - \frac{\sqrt{6}}{3} b = \frac{3 - \sqrt{6}}{3} b$. So, using the formula $V = \pi h^2 (r - h/3)$ from Exercise 7.2.27 with $r = b$, we find that the volume of each of the caps is $\pi\left(\frac{3-\sqrt{6}}{3} b\right)^2 \left(b - \frac{3-\sqrt{6}}{3\cdot 3} b\right) = \frac{15 - 6\sqrt{6}}{9} \cdot \frac{6+\sqrt{6}}{9} \pi b^3 = \left(\frac{2}{3} - \frac{7}{27}\sqrt{6}\right)\pi b^3$. So, using our first observation, the shared volume is $V = \frac{1}{2}\left(\frac{4}{3}\pi b^3\right) - 4\left(\frac{2}{3} - \frac{7}{27}\sqrt{6}\right)\pi b^3 = \left(\frac{28}{27}\sqrt{6} - 2\right)\pi b^3$.

17. We can assume that the cut is made along a vertical line $x = b > 0$, that the disk's boundary is the circle $x^2 + y^2 = 1$, and that the center of mass of the smaller piece (to the right of $x = b$) is $\left(\frac{1}{2}, 0\right)$. We wish to find b to two decimal places. We have



$\dfrac{1}{2} = \overline{x} = \dfrac{\int_b^1 x \cdot 2\sqrt{1-x^2}\,dx}{\int_b^1 2\sqrt{1-x^2}\,dx}$. Evaluating the numerator gives us

$-\int_b^1 (1-x^2)^{1/2}(-2x)\,dx = -\frac{2}{3}\left[\left(1-x^2\right)^{3/2}\right]_b^1 = -\frac{2}{3}\left[0 - \left(1-b^2\right)^{3/2}\right] = \frac{2}{3}(1-b^2)^{3/2}$. Using Formula 30 in the table of integrals, we find that the denominator is

$\left[x\sqrt{1-x^2} + \sin^{-1}x\right]_b^1 = \left(0 + \frac{\pi}{2}\right) - \left(b\sqrt{1-b^2} + \sin^{-1}b\right)$. Thus, we have

$\dfrac{1}{2} = \overline{x} = \dfrac{\frac{2}{3}(1-b^2)^{3/2}}{\frac{\pi}{2} - b\sqrt{1-b^2} - \sin^{-1}b}$, or, equivalently, $\frac{2}{3}(1-b^2)^{3/2} = \frac{\pi}{4} - \frac{1}{2}b\sqrt{1-b^2} - \frac{1}{2}\sin^{-1}b$. Solving this equation numerically with a calculator or CAS, we obtain $b \approx 0.138173$, or $b = 0.14$ m to two decimal places.

19. We use the Fundamental Theorem of Calculus to differentiate the given equation:

$[f(x)]^2 = 100 + \int_0^x \left\{[f(t)]^2 + [f'(t)]^2\right\} dt \quad\Rightarrow\quad 2f(x)f'(x) = [f(x)]^2 + [f'(x)]^2 \quad\Rightarrow$

$[f(x)]^2 + [f'(x)]^2 - 2f(x)f'(x) = 0 \quad\Rightarrow\quad [f(x) - f'(x)]^2 = 0 \quad\Leftrightarrow\quad f(x) = f'(x)$. We can solve this as a separable equation, or else use Theorem 3.4.2 with $k = 1$, which says that the solutions are $f(x) = Ce^x$. Now $[f(0)]^2 = 100$, so $f(0) = C = \pm 10$, and hence $f(x) = \pm 10e^x$ are the only functions satisfying the given equation.

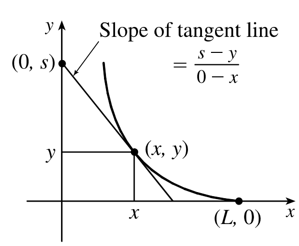
21. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{f(x)\,[f(h) - 1]}{h}$ [since $f(x+h) = f(x)f(h)$]

$= f(x) \lim\limits_{h\to 0} \dfrac{f(h) - 1}{h} = f(x) \lim\limits_{h\to 0} \dfrac{f(h) - f(0)}{h - 0} = f(x)f'(0) = f(x)$

Therefore, $f'(x) = f(x)$ for all x and from Theorem 3.4.2 we get $f(x) = Ae^x$. Now $f(0) = 1 \quad\Rightarrow$ $A = 1 \quad\Rightarrow\quad f(x) = e^x$.

23. (a) While running from $(L, 0)$ to (x, y), the dog travels a distance

$s = \displaystyle\int_x^L \sqrt{1 + (dy/dx)^2}\,dx = -\int_L^x \sqrt{1 + (dy/dx)^2}\,dx$, so $\dfrac{ds}{dx} = -\sqrt{1 + (dy/dx)^2}$. The dog and rabbit run at the same speed, so the rabbit's position when the dog has traveled a distance s is $(0, s)$. Since the dog runs straight for the rabbit, $\dfrac{dy}{dx} = \dfrac{s - y}{0 - x}$ (see the figure).



Thus, $s = y - x\,\dfrac{dy}{dx} \quad\Rightarrow\quad \dfrac{ds}{dx} = \dfrac{dy}{dx} - \left(x\,\dfrac{d^2y}{dx^2} + 1\,\dfrac{dy}{dx}\right) = -x\,\dfrac{d^2y}{dx^2}$. Equating the two expressions for $\dfrac{ds}{dx}$

gives us $x\,\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$, as claimed.

(b) Letting $z = \dfrac{dy}{dx}$, we obtain the differential equation $x\,\dfrac{dz}{dx} = \sqrt{1+z^2}$, or $\dfrac{dz}{\sqrt{1+z^2}} = \dfrac{dx}{x}$. Integrating:

$\ln x = \displaystyle\int \frac{dz}{\sqrt{1+z^2}} \overset{25}{=} \ln\left(z + \sqrt{1+z^2}\,\right) + C$. When $x = L$, $z = dy/dx = 0$, so $\ln L = \ln 1 + C$. Therefore,

$C = \ln L$, so $\ln x = \ln\left(\sqrt{1+z^2} + z\right) + \ln L = \ln\left[L\left(\sqrt{1+z^2} + z\right)\right] \quad\Rightarrow\quad x = L\left(\sqrt{1+z^2} + z\right) \quad\Rightarrow$

$\sqrt{1+z^2} = \dfrac{x}{L} - z \quad\Rightarrow\quad 1 + z^2 = \left(\dfrac{x}{L}\right)^2 - \dfrac{2xz}{L} + z^2 \quad\Rightarrow\quad \left(\dfrac{x}{L}\right)^2 - 2z\left(\dfrac{x}{L}\right) - 1 = 0 \quad\Rightarrow$

$z = \dfrac{(x/L)^2 - 1}{2(x/L)} = \dfrac{x^2 - L^2}{2Lx} = \dfrac{x}{2L} - \dfrac{L}{2}\dfrac{1}{x}$ [for $x > 0$]. Since $z = \dfrac{dy}{dx}$, $y = \dfrac{x^2}{4L} - \dfrac{L}{2}\ln x + C_1$. Since

$y = 0$ when $x = L$, $0 = \dfrac{L}{4} - \dfrac{L}{2}\ln L + C_1 \quad\Rightarrow\quad C_1 = \dfrac{L}{2}\ln L - \dfrac{L}{4}$. Thus,

$y = \dfrac{x^2}{4L} - \dfrac{L}{2}\ln x + \dfrac{L}{2}\ln L - \dfrac{L}{4} = \dfrac{x^2 - L^2}{4L} - \dfrac{L}{2}\ln\left(\dfrac{x}{L}\right)$.

(c) As $x \to 0^+$, $y \to \infty$, so the dog never catches the rabbit.

25. (a) We are given that $V = \frac{1}{3}\pi r^2 h$, $dV/dt = 60{,}000\pi$ ft^3/h, and $r = 1.5h = \frac{3}{2}h$. So $V = \frac{1}{3}\pi\left(\frac{3}{2}h\right)^2 h = \frac{3}{4}\pi h^3$

$\Rightarrow\quad \dfrac{dV}{dt} = \frac{3}{4}\pi \cdot 3h^2\,\dfrac{dh}{dt} = \frac{9}{4}\pi h^2\,\dfrac{dh}{dt}$. Therefore, $\dfrac{dh}{dt} = \dfrac{4(dV/dt)}{9\pi h^2} = \dfrac{240{,}000\pi}{9\pi h^2} = \dfrac{80{,}000}{3h^2}$ $(\star)$ $\quad\Rightarrow$

$\int 3h^2\,dh = \int 80{,}000\,dt \quad\Rightarrow\quad h^3 = 80{,}000t + C$. When $t = 0$, $h = 60$. Thus, $C = 60^3 = 216{,}000$, so

$h^3 = 80{,}000t + 216{,}000$. Let $h = 100$. Then $100^3 = 1{,}000{,}000 = 80{,}000t + 216{,}000 \quad\Rightarrow$

$80{,}000t = 784{,}000 \quad\Rightarrow\quad t = 9.8$, so the time required is 9.8 hours.

(b) The floor area of the silo is $F = \pi \cdot 200^2 = 40{,}000\pi$ ft^2, and the area of the base of the pile is

$A = \pi r^2 = \pi\left(\frac{3}{2}h\right)^2 = \frac{9\pi}{4}h^2$. So the area of the floor which is not covered when $h = 60$ is

$F - A = 40{,}000\pi - 8100\pi = 31{,}900\pi \approx 100{,}217$ ft^2. Now $A = \frac{9\pi}{4}h^2 \quad\Rightarrow\quad dA/dt = \frac{9\pi}{4}\cdot 2h\,(dh/dt)$,

and from $(\star)$ in part (a) we know that when $h = 60$, $dh/dt = \frac{80{,}000}{3(60)^2} = \frac{200}{27}$ ft/h. Therefore,

$dA/dt = \frac{9\pi}{4}(2)(60)\left(\frac{200}{27}\right) = 2000\pi \approx 6283$ ft^2/h.

(c) At $h = 90$ ft, $dV/dt = 60{,}000\pi - 20{,}000\pi = 40{,}000\pi$ ft^3/h. From $(\star)$ in part (a),

$\dfrac{dh}{dt} = \dfrac{4(dV/dt)}{9\pi h^2} = \dfrac{4(40{,}000\pi)}{9\pi h^2} = \dfrac{160{,}000}{9h^2} \quad\Rightarrow\quad \int 9h^2\,dh = \int 160{,}000\,dt \quad\Rightarrow\quad 3h^3 = 160{,}000t + C$.

When $t = 0$, $h = 90$; therefore, $C = 3 \cdot 729{,}000 = 2{,}187{,}000$. So $3h^3 = 160{,}000t + 2{,}187{,}000$. At the top,

$h = 100 \quad\Rightarrow\quad 3(100)^3 = 160{,}000t + 2{,}187{,}000 \quad\Rightarrow\quad t = \frac{813{,}000}{160{,}000} \approx 5.1$. The pile reaches the top after about

5.1 h.

27. Let $P(a, b)$ be any point on the curve. If m is the slope of the tangent line at P, then $m = y'(a)$, and an equation of the normal line at P is $y - b = -\dfrac{1}{m}(x - a)$, or equivalently, $y = -\dfrac{1}{m}x + b + \dfrac{a}{m}$.

The y-intercept is always 6, so $b + \dfrac{a}{m} = 6 \quad \Rightarrow \quad \dfrac{a}{m} = 6 - b \quad \Rightarrow \quad m = \dfrac{a}{6 - b}$.

We will solve the equivalent differential equation $\dfrac{dy}{dx} = \dfrac{x}{6 - y} \quad \Rightarrow \quad (6 - y)\,dy = x\,dx \quad \Rightarrow$

$\displaystyle\int (6 - y)\,dy = \int x\,dx \quad \Rightarrow \quad 6y - \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + C \quad \Rightarrow \quad 12y - y^2 = x^2 + K$. Since $(3, 2)$ is on the curve, $12(2) - 2^2 = 3^2 + K \quad \Rightarrow \quad K = 11$. So the curve is given by $12y - y^2 = x^2 + 11 \quad \Rightarrow$ $x^2 + y^2 - 12y + 36 = -11 + 36 \quad \Rightarrow \quad x^2 + (y - 6)^2 = 25$, a circle with center $(0, 6)$ and radius 5.