CHALLENGE PROBLEMS

CHAPTER 6

A Click here for answers.

S Click here for solutions.

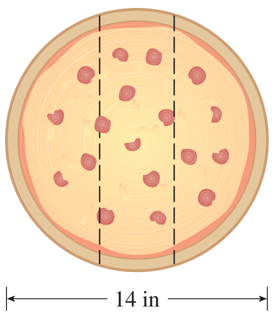


FIGURE FOR PROBLEM 1

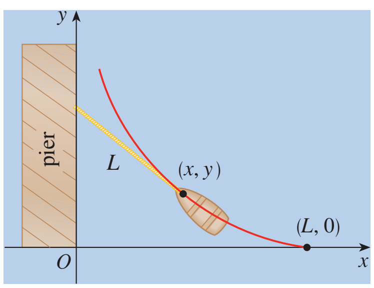


FIGURE FOR PROBLEM 4

1. Three mathematics students have ordered a 14-inch pizza. Instead of slicing it in the traditional way, they decide to slice it by parallel cuts, as shown in the figure. Being mathematics majors, they are able to determine where to slice so that each gets the same amount of pizza. Where are the cuts made?

2. Evaluate $\displaystyle\int \frac{1}{x^7 - x}\,dx$.

The straightforward approach would be to start with partial fractions, but that would be brutal. Try a substitution.

3. Evaluate $\displaystyle\int_0^1 \left(\sqrt[3]{1 - x^7} - \sqrt[7]{1 - x^3}\right) dx$.

4. A man initially standing at the point O walks along a pier pulling a rowboat by a rope of length L. The man keeps the rope straight and taut. The path followed by the boat is a curve called a *tractrix* and it has the property that the rope is always tangent to the curve (see the figure).

(a) Show that if the path followed by the boat is the graph of the function $y = f(x)$, then

$$f'(x) = \frac{dy}{dx} = \frac{-\sqrt{L^2 - x^2}}{x}$$

(b) Determine the function $y = f(x)$.

5. A function f is defined by

$$f(x) = \int_0^{\pi} \cos t \cos(x - t)\,dt \qquad 0 \leqslant x \leqslant 2\pi$$

Find the minimum value of f.

6. If n is a positive integer, prove that

$$\int_0^1 (\ln x)^n\,dx = (-1)^n n!$$

7. Show that

$$\int_0^1 (1 - x^2)^n\,dx = \frac{2^{2n}(n!)^2}{(2n + 1)!}$$

Hint: Start by showing that if I_n denotes the integral, then

$$I_{k+1} = \frac{2k + 2}{2k + 3} I_k$$

8. Suppose that f is a positive function such that f' is continuous.

(a) How is the graph of $y = f(x) \sin nx$ related to the graph of $y = f(x)$? What happens as $n \to \infty$?

(b) Make a guess as to the value of the limit

$$\lim_{n \to \infty} \int_0^1 f(x) \sin nx\,dx$$

based on graphs of the integrand.

(c) Using integration by parts, confirm the guess that you made in part (b). [Use the fact that, since f' is continuous, there is a constant M such that $|f'(x)| \leqslant M$ for $0 \leqslant x \leqslant 1$.]

9. If $0 < a < b$, find $\displaystyle\lim_{t \to 0} \left\{ \int_0^1 [bx + a(1 - x)]^t\,dx \right\}^{1/t}$.

10. Graph $f(x) = \sin(e^x)$ and use the graph to estimate the value of t such that $\int_t^{t+1} f(x)\,dx$ is a maximum. Then find the exact value of t that maximizes this integral.

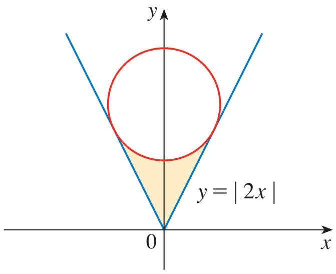


FIGURE FOR PROBLEM 11

11. The circle with radius 1 shown in the figure touches the curve $y = |2x|$ twice. Find the area of the region that lies between the two curves.

12. A rocket is fired straight up, burning fuel at the constant rate of b kilograms per second. Let $v = v(t)$ be the velocity of the rocket at time t and suppose that the velocity u of the exhaust gas is constant. Let $M = M(t)$ be the mass of the rocket at time t and note that M decreases as the fuel burns. If we neglect air resistance, it follows from Newton's Second Law that

$$F = M\frac{dv}{dt} - ub$$

where the force $F = -Mg$. Thus

$$\boxed{1} \qquad M\frac{dv}{dt} - ub = -Mg$$

Let M_1 be the mass of the rocket without fuel, M_2 the initial mass of the fuel, and $M_0 = M_1 + M_2$. Then, until the fuel runs out at time $t = M_2 b$, the mass is $M = M_0 - bt$.

(a) Substitute $M = M_0 - bt$ into Equation 1 and solve the resulting equation for v. Use the initial condition $v(0) = 0$ to evaluate the constant.
(b) Determine the velocity of the rocket at time $t = M_2/b$. This is called the *burnout velocity*.
(c) Determine the height of the rocket $y = y(t)$ at the burnout time.
(d) Find the height of the rocket at any time t.

13. Use integration by parts to show that, for all $x > 0$,

$$0 < \int_0^\infty \frac{\sin t}{\ln(1 + x + t)}\,dt < \frac{2}{\ln(1 + x)}$$

14. The **Chebyshev polynomials** T_n are defined by

$$T_n(x) = \cos(n \arccos x) \qquad n = 0, 1, 2, 3, \ldots$$

(a) What are the domain and range of these functions?
(b) We know that $T_0(x) = 1$ and $T_1(x) = x$. Express T_2 explicitly as a quadratic polynomial and T_3 as a cubic polynomial.
(c) Show that, for $n \geqslant 1$,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

(d) Use part (c) to show that T_n is a polynomial of degree n.
(e) Use parts (b) and (c) to express T_4, T_5, T_6, and T_7 explicitly as polynomials.
(f) What are the zeros of T_n? At what numbers does T_n have local maximum and minimum values?
(g) Graph T_2, T_3, T_4, and T_5 on a common screen.
(h) Graph T_5, T_6, and T_7 on a common screen.
(i) Based on your observations from parts (g) and (h), how are the zeros of T_n related to the zeros of T_{n+1}? What about the x-coordinates of the maximum and minimum values?
(j) Based on your graphs in parts (g) and (h), what can you say about $\int_{-1}^{1} T_n(x)\,dx$ when n is odd and when n is even?
(k) Use the substitution $u = \arccos x$ to evaluate the integral in part (j).
(l) The family of functions $f(x) = \cos(c \arccos x)$ are defined even when c is not an integer (but then f is not a polynomial). Describe how the graph of f changes as c increases.

ANSWERS

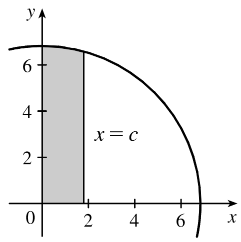
S Solutions

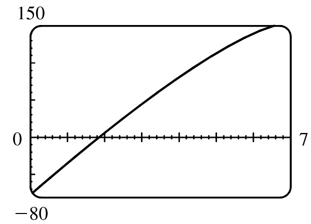
1. About 1.85 inches from the center **3.** 0 **5.** $f(\pi) = -\pi/2$ **9.** $(b^b a^{-a})^{1/(b-a)} e^{-1}$ **11.** $2 - \sin^{-1}(2/\sqrt{5})$

SOLUTIONS

E Exercises

1.

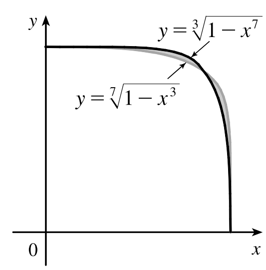




By symmetry, the problem can be reduced to finding the line $x = c$ such that the shaded area is one-third of the area of the quarter-circle. The equation of the circle is $y = \sqrt{49 - x^2}$, so we require that $\int_0^c \sqrt{49 - x^2}\,dx = \frac{1}{3} \cdot \frac{1}{4}\pi(7)^2$ $\Leftrightarrow$ $\left[\frac{1}{2}x\sqrt{49 - x^2} + \frac{49}{2}\sin^{-1}(x/7)\right]_0^c = \frac{49}{12}\pi$ [by Formula 30] $\Leftrightarrow$ $\frac{1}{2}c\sqrt{49 - c^2} + \frac{49}{2}\sin^{-1}(c/7) = \frac{49}{12}\pi$.

This equation would be difficult to solve exactly, so we plot the left-hand side as a function of c, and find that the equation holds for $c \approx 1.85$. So the cuts should be made at distances of about 1.85 inches from the center of the pizza.

3. The given integral represents the difference of the shaded areas, which appears to be 0. It can be calculated by integrating with respect to either x or y, so we find x in terms of y for each curve: $y = \sqrt[3]{1 - x^7}$ $\Rightarrow$ $x = \sqrt[7]{1 - y^3}$ and $y = \sqrt[7]{1 - x^3}$ $\Rightarrow$ $x = \sqrt[3]{1 - y^7}$, so $\int_0^1 \left(\sqrt[3]{1 - y^7} - \sqrt[7]{1 - y^3}\right) dy = \int_0^1 \left(\sqrt[7]{1 - x^3} - \sqrt[3]{1 - x^7}\right) dx$. But this equation is of the form $z = -z$. So $\int_0^1 \left(\sqrt[3]{1 - x^7} - \sqrt[7]{1 - x^3}\right) dx = 0$.



5. Recall that $\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$. So

$$f(x) = \int_0^\pi \cos t \cos(x - t)\,dt = \frac{1}{2}\int_0^\pi [\cos(t + x - t) + \cos(t - x + t)]\,dt = \frac{1}{2}\int_0^\pi [\cos x + \cos(2t - x)]\,dt$$

$$= \frac{1}{2}\left[t\cos x + \frac{1}{2}\sin(2t - x)\right]_0^\pi = \frac{\pi}{2}\cos x + \frac{1}{4}\sin(2\pi - x) - \frac{1}{4}\sin(-x)$$

$$= \frac{\pi}{2}\cos x + \frac{1}{4}\sin(-x) - \frac{1}{4}\sin(-x) = \frac{\pi}{2}\cos x$$

The minimum of $\cos x$ on this domain is -1, so the minimum value of $f(x)$ is $f(\pi) = -\frac{\pi}{2}$.

7. In accordance with the hint, we let $I_k = \int_0^1 \left(1 - x^2\right)^k dx$, and we find an expression for I_{k+1} in terms of I_k. We integrate I_{k+1} by parts with $u = \left(1 - x^2\right)^{k+1}$ $\Rightarrow$ $du = (k+1)\left(1 - x^2\right)^k(-2x)$, $dv = dx$ $\Rightarrow$ $v = x$, and then split the remaining integral into identifiable quantities:

$$I_{k+1} = x\left(1 - x^2\right)^{k+1}\Big|_0^1 + 2(k+1)\int_0^1 x^2\left(1 - x^2\right)^k dx$$

$$= (2k+2)\int_0^1 \left(1 - x^2\right)^k\left[1 - \left(1 - x^2\right)\right] dx = (2k+2)(I_k - I_{k+1})$$

So $I_{k+1}[1+(2k+2)] = (2k+2)I_k \quad \Rightarrow \quad I_{k+1} = \dfrac{2k+2}{2k+3} I_k$. Now to complete the proof, we use induction:

$I_0 = 1 = \dfrac{2^0 (0!)^2}{1!}$, so the formula holds for $n = 0$. Now suppose it holds for $n = k$. Then

$$I_{k+1} = \frac{2k+2}{2k+3} I_k = \frac{2k+2}{2k+3}\left[\frac{2^{2k}(k!)^2}{(2k+1)!}\right] = \frac{2(k+1)2^{2k}(k!)^2}{(2k+3)(2k+1)!} = \frac{2(k+1)}{2k+2} \cdot \frac{2(k+1)2^{2k}(k!)^2}{(2k+3)(2k+1)!}$$
$$= \frac{[2(k+1)]^2 2^{2k}(k!)^2}{(2k+3)(2k+2)(2k+1)!} = \frac{2^{2(k+1)}[(k+1)!]^2}{[2(k+1)+1]!}$$

So by induction, the formula holds for all integers $n \geq 0$.

9. $0 < a < b$. Now

$$\int_0^1 [bx + a(1-x)]^t \, dx = \int_a^b \tfrac{u^t}{(b-a)} \, du \quad \text{[put } u = bx + a(1-x)\text{]} \quad = \left[\frac{u^{t+1}}{(t+1)(b-a)}\right]_a^b = \frac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}.$$

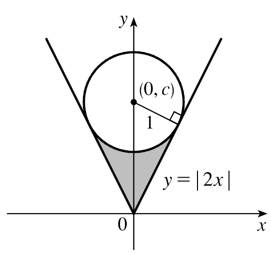
Now let $y = \lim\limits_{t \to 0} \left[\dfrac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}\right]^{1/t}$. Then $\ln y = \lim\limits_{t \to 0} \left[\dfrac{1}{t} \ln \dfrac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}\right]$. This limit is of the form $0/0$,

so we can apply l'Hospital's Rule to get

$$\ln y = \lim_{t \to 0} \left[\frac{b^{t+1} \ln b - a^{t+1} \ln a}{b^{t+1} - a^{t+1}} - \frac{1}{t+1}\right] = \frac{b \ln b - a \ln a}{b-a} - 1 = \frac{b \ln b}{b-a} - \frac{a \ln a}{b-a} - \ln e = \ln \frac{b^{b/(b-a)}}{e a^{a/(b-a)}}.$$

Therefore, $y = e^{-1}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}$.

11.



An equation of the circle with center $(0, c)$ and radius 1 is $x^2 + (y-c)^2 = 1^2$, so an equation of the lower semicircle is $y = c - \sqrt{1-x^2}$. At the points of tangency, the slopes of the line and semicircle must be equal. For $x \geq 0$, we must have

$$y' = 2 \quad \Rightarrow \quad \frac{x}{\sqrt{1-x^2}} = 2 \quad \Rightarrow \quad x = 2\sqrt{1-x^2} \quad \Rightarrow$$
$$x^2 = 4(1-x^2) \quad \Rightarrow \quad 5x^2 = 4 \quad \Rightarrow \quad x^2 = \tfrac{4}{5} \quad \Rightarrow \quad x = \tfrac{2}{5}\sqrt{5}$$

and so $y = 2\left(\frac{2}{5}\sqrt{5}\right) = \frac{4}{5}\sqrt{5}$. The slope of the perpendicular line segment is $-\frac{1}{2}$, so an equation of the line segment is $y - \frac{4}{5}\sqrt{5} = -\frac{1}{2}\left(x - \frac{2}{5}\sqrt{5}\right) \quad \Leftrightarrow \quad y = -\frac{1}{2}x + \frac{1}{5}\sqrt{5} + \frac{4}{5}\sqrt{5} \quad \Leftrightarrow \quad y = -\frac{1}{2}x + \sqrt{5}$, so $c = \sqrt{5}$ and an equation of the lower semicircle is $y = \sqrt{5} - \sqrt{1-x^2}$. Thus, the shaded area is

$$2\int_0^{(2/5)\sqrt{5}} \left[\left(\sqrt{5} - \sqrt{1-x^2}\right) - 2x\right] dx \stackrel{30}{=} 2\left[\sqrt{5}\,x - \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1} x - x^2\right]_0^{(2/5)\sqrt{5}}$$
$$= 2\left[2 - \frac{\sqrt{5}}{5} \cdot \frac{1}{\sqrt{5}} - \frac{1}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - \frac{4}{5}\right] - 2(0)$$
$$= 2\left[1 - \frac{1}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right] = 2 - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

13. We integrate by parts with $u = \dfrac{1}{\ln(1+x+t)}$, $dv = \sin t\,dt$, so $du = \dfrac{-1}{(1+x+t)[\ln(1+x+t)]^2}$ and $v = -\cos t$. The integral becomes

$$I = \int_0^\infty \frac{\sin t\,dt}{\ln(1+x+t)} = \lim_{b\to\infty} \left(\left[\frac{-\cos t}{\ln(1+x+t)} \right]_0^b - \int_0^b \frac{\cos t\,dt}{(1+x+t)[\ln(1+x+t)]^2} \right)$$

$$= \lim_{b\to\infty} \frac{-\cos b}{\ln(1+x+b)} + \frac{1}{\ln(1+x)} + \int_0^\infty \frac{-\cos t\,dt}{(1+x+t)[\ln(1+x+t)]^2} = \frac{1}{\ln(1+x)} + J$$

where $J = \displaystyle\int_0^\infty \frac{-\cos t\,dt}{(1+x+t)[\ln(1+x+t)]^2}$. Now $-1 \le -\cos t \le 1$ for all t; in fact, the inequality is strict except at isolated points. So $-\displaystyle\int_0^\infty \frac{dt}{(1+x+t)[\ln(1+x+t)]^2} < J < \int_0^\infty \frac{dt}{(1+x+t)[\ln(1+x+t)]^2} \quad\Leftrightarrow$

$-\dfrac{1}{\ln(1+x)} < J < \dfrac{1}{\ln(1+x)} \quad\Leftrightarrow\quad 0 < I < \dfrac{2}{\ln(1+x)}$.