

## 1.1 FUNCTIONS AND THEIR REPRESENTATIONS

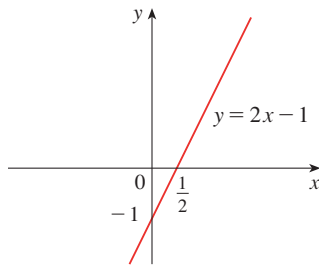


FIGURE 1

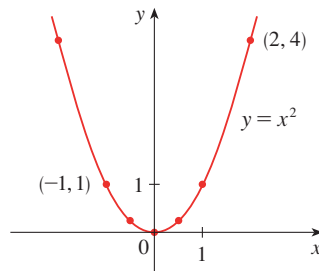


FIGURE 2

**EXAMPLE A** Sketch the graph and find the domain and range of each function.

(a)  $f(x) = 2x - 1$

(b)  $g(x) = x^2$

**SOLUTION**

(a) The equation of the graph is  $y = 2x - 1$ , and we recognize this as being the equation of a line with slope 2 and  $y$ -intercept  $-1$ . (Recall the slope-intercept form of the equation of a line:  $y = mx + b$ .) This enables us to sketch the graph of  $f$  in Figure 1. The expression  $2x - 1$  is defined for all real numbers, so the domain of  $f$  is the set of all real numbers, which we denote by  $\mathbb{R}$ . The graph shows that the range is also  $\mathbb{R}$ .

(b) Since  $g(2) = 2^2 = 4$  and  $g(-1) = (-1)^2 = 1$ , we could plot the points  $(2, 4)$  and  $(-1, 1)$ , together with a few other points on the graph, and join them to produce the graph (Figure 2). The equation of the graph is  $y = x^2$ , which represents a parabola. The domain of  $g$  is  $\mathbb{R}$ . The range of  $g$  consists of all values of  $g(x)$ , that is, all numbers of the form  $x^2$ . But  $x^2 \geq 0$  for all numbers  $x$  and any positive number  $y$  is a square. So the range of  $g$  is  $\{y \mid y \geq 0\} = [0, \infty)$ . This can also be seen from Figure 2. ■

**EXAMPLE B** If  $f(x) = 2x^2 - 5x + 1$  and  $h \neq 0$ , evaluate  $\frac{f(a+h) - f(a)}{h}$ .

**SOLUTION** We first evaluate  $f(a+h)$  by replacing  $x$  by  $a+h$  in the expression for  $f(x)$ :

$$\begin{aligned} f(a+h) &= 2(a+h)^2 - 5(a+h) + 1 \\ &= 2(a^2 + 2ah + h^2) - 5(a+h) + 1 \\ &= 2a^2 + 4ah + 2h^2 - 5a - 5h + 1 \end{aligned}$$

Then we substitute into the given expression and simplify:

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(2a^2 + 4ah + 2h^2 - 5a - 5h + 1) - (2a^2 - 5a + 1)}{h} \\ &= \frac{2a^2 + 4ah + 2h^2 - 5a - 5h + 1 - 2a^2 + 5a - 1}{h} \\ &= \frac{4ah + 2h^2 - 5h}{h} = 4a + 2h - 5 \end{aligned}$$

$t$	$C(t)$
0	0.0800
2	0.0570
4	0.0408
6	0.0295
8	0.0210

**EXAMPLE C** The data shown at the left come from an experiment on the lactonization of hydroxyvaleric acid at 25°C. They give the concentration  $C(t)$  of this acid (in moles per liter) after  $t$  minutes. Use these data to draw an approximation to the graph of the concentration function. Then use this graph to estimate the concentration after 5 minutes.

**SOLUTION** We plot the five points corresponding to the data from the table in Figure 3. The data points look quite well behaved, so we simply draw a smooth curve through them by hand as in Figure 4.

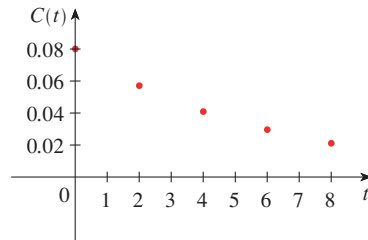


FIGURE 3

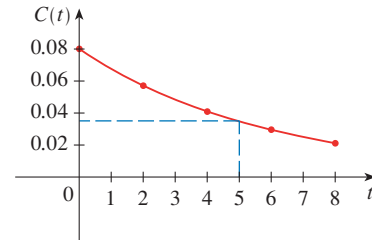


FIGURE 4

Then we use the graph to estimate that the concentration after 5 minutes is

$$C(5) \approx 0.035 \text{ mole/liter.}$$

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**EXAMPLE D** A rectangular storage container with an open top has a volume of  $10 \text{ m}^3$ . The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.

**SOLUTION** We draw a diagram as in Figure 5 and introduce notation by letting  $w$  and  $2w$  be the width and length of the base, respectively, and  $h$  be the height.

The area of the base is  $(2w)w = 2w^2$ , so the cost, in dollars, of the material for the base is  $10(2w^2)$ . Two of the sides have area  $wh$  and the other two have area  $2wh$ , so the cost of the material for the sides is  $6[2(wh) + 2(2wh)]$ . The total cost is therefore

$$C = 10(2w^2) + 6[2(wh) + 2(2wh)] = 20w^2 + 36wh$$

To express  $C$  as a function of  $w$  alone, we need to eliminate  $h$  and we do so by using the fact that the volume is  $10 \text{ m}^3$ . Thus

$$w(2w)h = 10$$

which gives 
$$h = \frac{10}{2w^2} = \frac{5}{w^2}$$

Substituting this into the expression for  $C$ , we have

$$C = 20w^2 + 36w\left(\frac{5}{w^2}\right) = 20w^2 + \frac{180}{w}$$

Therefore, the equation

$$C(w) = 20w^2 + \frac{180}{w} \quad w > 0$$

expresses  $C$  as a function of  $w$ .

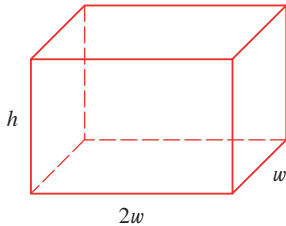
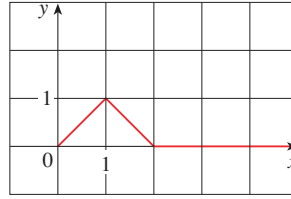


FIGURE 5

**EXAMPLE E** Find a formula for the function  $f$  graphed in Figure 6.



**FIGURE 6**

**SOLUTION** The line through  $(0, 0)$  and  $(1, 1)$  has slope  $m = 1$  and  $y$ -intercept  $b = 0$ , so its equation is  $y = x$ . Thus, for the part of the graph of  $f$  that joins  $(0, 0)$  to  $(1, 1)$ , we have

$$f(x) = x \quad \text{if } 0 \leq x \leq 1$$

The line through  $(1, 1)$  and  $(2, 0)$  has slope  $m = -1$ , so its point-slope form is

- Point-slope form of the equation of a line:

$$y - y_1 = m(x - x_1)$$

$$y - 0 = (-1)(x - 2) \quad \text{or} \quad y = 2 - x$$

So we have

$$f(x) = 2 - x \quad \text{if } 1 < x \leq 2$$

We also see that the graph of  $f$  coincides with the  $x$ -axis for  $x > 2$ . Putting this information together, we have the following three-piece formula for  $f$ :

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } 1 < x \leq 2 \\ 0 & \text{if } x > 2 \end{cases}$$