2.4 THE PRODUCT AND QUOTIENT RULES

EXAMPLE A If $f(x) = \sqrt{x}\, g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

SOLUTION Applying the Product Rule, we get

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\sqrt{x}\, g(x)\right] = \sqrt{x}\,\frac{d}{dx}[g(x)] + g(x)\,\frac{d}{dx}\left[\sqrt{x}\right] \\ &= \sqrt{x}\, g'(x) + g(x)\cdot \tfrac{1}{2}x^{-1/2} \\ &= \sqrt{x}\, g'(x) + \frac{g(x)}{2\sqrt{x}} \end{aligned}$$

So $$f'(4) = \sqrt{4}\, g'(4) + \frac{g(4)}{2\sqrt{4}} = 2\cdot 3 + \frac{2}{2\cdot 2} = 6.5$$ ■

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V **EXAMPLE B** A telephone company wants to estimate the number of new residential phone lines that it will need to install during the upcoming month. At the beginning of January the company had 100,000 subscribers, each of whom had 1.2 phone lines, on average. The company estimated that its subscribership was increasing at the rate of 1000 monthly. By polling its existing subscribers, the company found that each intended to install an average of 0.01 new phone lines by the end of January. Estimate the number of new lines the company will have to install in January by calculating the rate of increase of lines at the beginning of the month.

SOLUTION Let $s(t)$ be the number of subscribers and let $n(t)$ be the number of phone lines per subscriber at time t, where t is measured in months and $t = 0$ corresponds to the beginning of January. Then the total number of lines is given by

$$L(t) = s(t)n(t)$$

and we want to find $L'(0)$. According to the Product Rule, we have

$$L'(t) = \frac{d}{dt}[s(t)n(t)] = s(t)\,\frac{d}{dt}\,n(t) + n(t)\,\frac{d}{dt}\,s(t)$$

We are given that $s(0) = 100{,}000$ and $n(0) = 1.2$. The company's estimates concerning rates of increase are that $s'(0) \approx 1000$ and $n'(0) \approx 0.01$. Therefore,

$$\begin{aligned} L'(0) &= s(0)n'(0) + n(0)s'(0) \\ &\approx 100{,}000\cdot 0.01 + 1.2\cdot 1000 = 2200 \end{aligned}$$

The company will need to install approximately 2200 new phone lines in January.

Notice that the two terms arising from the Product Rule come from different sources—old subscribers and new subscribers. One contribution to L' is the number of existing subscribers (100,000) times the rate at which they order new lines (about 0.01 per subscriber monthly). A second contribution is the average number of lines per subscriber (1.2 at the beginning of the month) times the rate of increase of subscribers (1000 monthly). ■