6.3 PARTIAL FRACTIONS

EXAMPLE A Find $\int \frac{dx}{x^2 - a^2}$, where $a \neq 0$.

SOLUTION The method of partial fractions gives

$$\frac{1}{x^2 - a^2} = \frac{1}{(x - a)(x + a)} = \frac{A}{x - a} + \frac{B}{x + a}$$

and therefore

$$A(x + a) + B(x - a) = 1$$

Using the method of the note after Example 2, we put $x = a$ in this equation and get $A(2a) = 1$, so $A = 1/(2a)$. If we put $x = -a$, we get $B(-2a) = 1$, so $B = -1/(2a)$. Thus

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \int \left(\frac{1}{x - a} - \frac{1}{x + a} \right) dx$$

$$= \frac{1}{2a} \left(\ln |x - a| - \ln |x + a| \right) + C$$

Since $\ln x - \ln y = \ln(x/y)$, we can write the integral as

I $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$$ ■

EXAMPLE B Evaluate $\int \frac{\sqrt{x + 4}}{x}\, dx$.

SOLUTION Let $u = \sqrt{x + 4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2u\, du$. Therefore

$$\int \frac{\sqrt{x + 4}}{x}\, dx = \int \frac{u}{u^2 - 4}\, 2u\, du = 2 \int \frac{u^2}{u^2 - 4}\, du$$

$$= 2 \int \left(1 + \frac{4}{u^2 - 4} \right) du$$

We can evaluate this integral either by factoring $u^2 - 4$ as $(u - 2)(u + 2)$ and using partial fractions or by using Formula 1 in Example A with $a = 2$:

$$\int \frac{\sqrt{x + 4}}{x}\, dx = 2 \int du + 8 \int \frac{du}{u^2 - 4}$$

$$= 2u + 8 \cdot \frac{1}{2 \cdot 2} \ln \left| \frac{u - 2}{u + 2} \right| + C$$

$$= 2\sqrt{x + 4} + 2 \ln \left| \frac{\sqrt{x + 4} - 2}{\sqrt{x + 4} + 2} \right| + C$$ ■