Many differential equations can’t be solved explicitly in terms of finite combinations of simple familiar functions. This is true even for a simple-looking equation like

$$y'' - 2xy' + y = 0$$

But it is important to be able to solve equations such as Equation 1 because they arise from physical problems and, in particular, in connection with the Schrödinger equation in quantum mechanics. In such a case we use the method of power series; that is, we look for a solution of the form

$$y = f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

The method is to substitute this expression into the differential equation and determine the values of the coefficients $c_0, c_1, c_2, \ldots$.

Before using power series to solve Equation 1, we illustrate the method on the simpler equation in Example 1.

**EXAMPLE 1** Use power series to solve the equation $y'' + y = 0$.

**SOLUTION** We assume there is a solution of the form

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

We can differentiate power series term by term, so

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = 2c_2 + 2 \cdot 3c_3 x + \cdots = \sum_{n=2}^{\infty} n(n - 1)c_n x^{n-2}$$

In order to compare the expressions for $y$ and $y''$ more easily, we rewrite $y''$ as follows:

$$y'' = \sum_{n=0}^{\infty} (n + 2)(n + 1)c_{n+2} x^n$$

Substituting the expressions in Equations 2 and 4 into the differential equation, we obtain

$$\sum_{n=0}^{\infty} (n + 2)(n + 1)c_{n+2} x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} [(n + 2)(n + 1)c_{n+2} + c_n] x^n = 0$$

If two power series are equal, then the corresponding coefficients must be equal. Therefore, the coefficients of $x^n$ in Equation 5 must be 0:

$$(n + 2)(n + 1)c_{n+2} + c_n = 0$$
Equation 6 is called a recursion relation. If $c_0$ and $c_1$ are known, this equation allows us to determine the remaining coefficients recursively by putting $n = 0, 1, 2, 3, \ldots$ in succession.

\[ c_{n+2} = -\frac{c_n}{(n+1)(n+2)} \quad n = 0, 1, 2, 3, \ldots \]

By now we see the pattern:

Put $n = 0$: \[ c_2 = -\frac{c_0}{1 \cdot 2} \]

Put $n = 1$: \[ c_3 = -\frac{c_1}{2 \cdot 3} \]

Put $n = 2$: \[ c_4 = -\frac{c_2}{3 \cdot 4} = -\frac{c_0}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{c_0}{4!} \]

Put $n = 3$: \[ c_5 = -\frac{c_3}{4 \cdot 5} = -\frac{c_1}{2 \cdot 3 \cdot 4 \cdot 5} = \frac{c_1}{5!} \]

Put $n = 4$: \[ c_6 = -\frac{c_4}{5 \cdot 6} = -\frac{c_0}{4! \cdot 5 \cdot 6} = -\frac{c_0}{6!} \]

Put $n = 5$: \[ c_7 = -\frac{c_5}{6 \cdot 7} = -\frac{c_1}{5! \cdot 6 \cdot 7} = -\frac{c_1}{7!} \]

For the even coefficients, \[ c_{2n} = (-1)^n \frac{c_0}{(2n)!} \]

For the odd coefficients, \[ c_{2n+1} = (-1)^n \frac{c_1}{(2n + 1)!} \]

Putting these values back into Equation 2, we write the solution as

\[
y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots
\]

\[
= c_0 \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \right)
\]

\[
+ c_1 \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n + 1)!} + \cdots \right)
\]

\[
= c_0 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n + 1)!}
\]

Notice that there are two arbitrary constants, $c_0$ and $c_1$.

**NOTE 1** We recognize the series obtained in Example 1 as being the Maclaurin series for $\cos x$ and $\sin x$. (See Equations 8.7.17 and 8.7.16.) Therefore, we could write the solution as

\[ y(x) = c_0 \cos x + c_1 \sin x \]

But we are not usually able to express power series solutions of differential equations in terms of known functions.
**EXAMPLE 2** Solve \( y'' - 2xy' + y = 0. \)

**SOLUTION** We assume there is a solution of the form

\[
y = \sum_{n=0}^{\infty} c_n x^n
\]

Then

\[
y' = \sum_{n=1}^{\infty} nc_n x^{n-1}
\]

and

\[
y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n
\]

as in Example 1. Substituting in the differential equation, we get

\[
\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - 2x \sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0
\]

\[
\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - \sum_{n=1}^{\infty} 2nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0
\]

\[
\sum_{n=1}^{\infty} 2nc_n x^n = \sum_{n=0}^{\infty} 2nc_n x^n
\]

\[
\sum_{n=0}^{\infty} [(n+2)(n+1)c_{n+2} - (2n - 1)c_n] x^n = 0
\]

This equation is true if the coefficient of \( x^n \) is 0:

\[
(n + 2)(n + 1)c_{n+2} - (2n - 1)c_n = 0
\]

\[
c_{n+2} = \frac{2n + 1}{(n + 1)(n + 2)} c_n \quad n = 0, 1, 2, 3, \ldots
\]

We solve this recursion relation by putting \( n = 0, 1, 2, 3, \ldots \) successively in Equation 7:

Put \( n = 0 \):
\[
c_2 = -\frac{1}{1 \cdot 2} c_0
\]

Put \( n = 1 \):
\[
c_3 = \frac{1}{2 \cdot 3} c_1
\]

Put \( n = 2 \):
\[
c_4 = \frac{3}{3 \cdot 4} c_2 = -\frac{3}{1 \cdot 2 \cdot 3 \cdot 4} c_0 = -\frac{3}{4!} c_0
\]

Put \( n = 3 \):
\[
c_5 = \frac{5}{4 \cdot 5} c_3 = \frac{1 \cdot 5}{2 \cdot 3 \cdot 4 \cdot 5} c_1 = \frac{1 \cdot 5}{5!} c_1
\]

Put \( n = 4 \):
\[
c_6 = \frac{7}{5 \cdot 6} c_4 = -\frac{3 \cdot 7}{4! \cdot 5 \cdot 6} c_0 = -\frac{3 \cdot 7}{6!} c_0
\]

Put \( n = 5 \):
\[
c_7 = \frac{9}{6 \cdot 7} c_5 = \frac{1 \cdot 5 \cdot 9}{5! \cdot 6 \cdot 7} c_1 = \frac{1 \cdot 5 \cdot 9}{7!} c_1
\]

Put \( n = 6 \):
\[
c_8 = \frac{11}{7 \cdot 8} c_6 = -\frac{3 \cdot 7 \cdot 11}{8!} c_0
\]

Put \( n = 7 \):
\[
c_9 = \frac{13}{8 \cdot 9} c_7 = \frac{1 \cdot 5 \cdot 9 \cdot 13}{9!} c_1
\]
In general, the even coefficients are given by
\[ c_{2n} = \frac{3 \cdot 7 \cdot 11 \cdots (4n - 5)}{(2n)!} c_0 \]
and the odd coefficients are given by
\[ c_{2n+1} = \frac{1 \cdot 5 \cdot 9 \cdots (4n - 3)}{(2n + 1)!} c_1 \]
The solution is
\[ y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \]
\[ = c_0 \left( 1 - \frac{1}{2!} x^2 - \frac{3}{4!} x^4 - \frac{3 \cdot 7}{6!} x^6 - \frac{3 \cdot 7 \cdot 11}{8!} x^8 - \cdots \right) \]
\[ + c_1 \left( x + \frac{1}{3!} x^3 + \frac{1 \cdot 5}{5!} x^5 + \frac{1 \cdot 5 \cdot 9}{7!} x^7 + \frac{1 \cdot 5 \cdot 9 \cdot 13}{9!} x^9 + \cdots \right) \]
or
\[ y = c_0 \left( 1 - \frac{1}{2!} x^2 - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n - 5)}{(2n)!} x^{2n} \right) \]
\[ + c_1 \left( x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdots (4n - 3)}{(2n + 1)!} x^{2n+1} \right) \]

**NOTE 2** In Example 2 we had to assume that the differential equation had a series solution. But now we could verify directly that the function given by Equation 8 is indeed a solution.

**NOTE 3** Unlike the situation of Example 1, the power series that arise in the solution of Example 2 do not define elementary functions. The functions
\[ y_1(x) = 1 - \frac{1}{2!} x^2 - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n - 5)}{(2n)!} x^{2n} \]
and
\[ y_2(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdots (4n - 3)}{(2n + 1)!} x^{2n+1} \]
are perfectly good functions but they can’t be expressed in terms of familiar functions. We can use these power series expressions for \( y_1 \) and \( y_2 \) to compute approximate values of the functions and even to graph them. Figure 1 shows the first few partial sums \( T_0, T_2, T_4, \ldots \) (Taylor polynomials) for \( y_1(x) \), and we see how they converge to \( y_1 \). In this way we can graph both \( y_1 \) and \( y_2 \) in Figure 2.

**NOTE 4** If we were asked to solve the initial-value problem
\[ y'' - 2xy' + y = 0 \quad y(0) = 0 \quad y'(0) = 1 \]
we would observe that
\[ c_0 = y(0) = 0 \quad c_1 = y'(0) = 1 \]
This would simplify the calculations in Example 2, since all of the even coefficients would be 0. The solution to the initial-value problem is
\[ y(x) = x + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdots \cdot (4n - 3)}{(2n + 1)!} x^{2n+1} \]
EXERCISES

1–11 Use power series to solve the differential equation.

1. \( y' - y = 0 \)
2. \( y' = xy \)
3. \( y' = x^2y \)
4. \( (x - 3)y' + 2y = 0 \)
5. \( y'' + xy' + y = 0 \)
6. \( y'' = y \)
7. \( (x^2 + 1)y'' + xy' - y = 0 \)
8. \( y'' = xy \)
9. \( y'' - xy' - y = 0, \ y(0) = 1, \ y'(0) = 0 \)
10. \( y'' + x^2y = 0, \ y(0) = 1, \ y'(0) = 0 \)
11. \( y'' + x^2y' + xy = 0, \ y(0) = 0, \ y'(0) = 1 \)
12. The solution of the initial-value problem
   \[ x^2y'' + xy' + x^2y = 0 \quad y(0) = 1 \quad y'(0) = 0 \]
   is called a Bessel function of order 0.
   (a) Solve the initial-value problem to find a power series expansion for the Bessel function.
   (b) Graph several Taylor polynomials until you reach one that looks like a good approximation to the Bessel function on the interval \([-5, 5]\).
ANSWERS

1. \( c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = c_0 e^x \)

3. \( c_0 \sum_{n=0}^{\infty} \frac{x^{3n}}{3^n n!} = c_0 e^{x^3/3} \)

5. \( c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n + 1)!} x^{2n+1} \)

7. \( c_0 + c_1 x + c_0 \frac{x^2}{2} + c_0 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(2n - 3)!}{2^{2n-2}n!(n - 2)!} x^{2n} \)

9. \( \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = e^{x^2/2} \)

11. \( x + \sum_{n=1}^{\infty} \frac{(-1)^n 2^5 5^2 \cdots (3n - 1)^2}{(3n + 1)!} x^{3n+1} \)
1. Let \( y(x) = \sum_{n=0}^{\infty} c_n x^n \). Then \( y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \) and the given equation, \( y' - y = 0 \), becomes

\[
\sum_{n=0}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0. \text{ Replacing } n \text{ by } n+1 \text{ in the first sum gives } \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=0}^{\infty} c_n x^n = 0, \text{ so}
\]

\[
\sum_{n=0}^{\infty} [(n+1) c_{n+1} - c_n] x^n = 0. \text{ Equating coefficients gives } (n+1) c_{n+1} - c_n = 0, \text{ so the recursion relation is}
\]

\[
c_{n+1} = \frac{c_n}{n+1}, \quad n = 0, 1, 2, \ldots \text{. Then } c_1 = c_0, \quad c_2 = \frac{1}{2} c_0, \quad c_3 = \frac{1}{3} \frac{1}{2} c_0, \quad c_4 = \frac{1}{4} \frac{1}{3} \frac{1}{2} c_0, \quad c_5 = \frac{1}{5} \frac{1}{4} \frac{1}{3} \frac{1}{2} c_0, \text{ and}
\]

in general, \( c_n = \frac{c_0}{n!} \). Thus, the solution is \( y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{c_0}{n!} x^n = c_0 e^x \).

3. Assuming \( y(x) = \sum_{n=0}^{\infty} c_n x^n \), we have \( y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n \) and

\[
-x^2 y = -\sum_{n=0}^{\infty} c_n x^{n+2} = -\sum_{n=2}^{\infty} c_{n-2} x^n. \text{ Hence, the equation } y' = x^2 y \text{ becomes}
\]

\[
\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=2}^{\infty} c_{n-2} x^n = 0 \text{ or } c_1 + 2 c_2 x + \sum_{n=2}^{\infty} [(n+1) c_{n+1} - c_{n-2}] x^n = 0. \text{ Equating coefficients gives}
\]

\[
c_1 = c_2 = 0 \text{ and } c_{n+1} = \frac{c_n}{n+1} \text{ for } n = 2, 3, \ldots \text{. But } c_1 = 0, \text{ so } c_4 = 0 \text{ and}
\]

\[
c_7 = 0 \text{ and in general } c_{3 n+1} = 0. \text{ Similarly } c_2 = 0 \text{ so } c_{3 n+2} = 0. \text{ Finally } c_3 = \frac{c_0}{3}.
\]

\[
c_0 = \frac{c_3}{6} = \frac{c_0}{6 \cdot 3} = \frac{c_0}{3!} \text{ and } c_3 = \frac{c_0}{3 \cdot 3!} = \frac{c_0}{3^2 \cdot 3!}, \ldots \text{, and } c_{3 n} = \frac{c_0}{3^n \cdot n!} \text{. Thus, the solution is}
\]

\[
y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{c_0}{3^n \cdot n!} x^3 n = c_0 \sum_{n=0}^{\infty} \frac{x^{3 n}}{3^n \cdot n!} = c_0 \sum_{n=0}^{\infty} \frac{(x^3/3)^n}{n!} = c_0 e^{x^3/3}.
\]

5. Let \( y(x) = \sum_{n=0}^{\infty} c_n x^n \) \( \Rightarrow \) \( y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \) and \( y''(x) = \sum_{n=0}^{\infty} (n+2) (n+1) c_{n+2} x^n \). The differential equation becomes \( \sum_{n=1}^{\infty} (n+2) (n+1) c_{n+2} x^n + x \sum_{n=0}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0 \) or

\[
\sum_{n=0}^{\infty} [(n+2) (n+1) c_{n+2} + n c_n + c_n] x^n \quad \text{ (since } \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} n c_n x^n \text{). Equating coefficients gives}
\]

\[
(n+2) (n+1) c_{n+2} + (n+1) c_n = 0 \text{, thus the recursion relation is } c_{n+2} = \frac{-(n+1) c_n}{(n+2)(n+1)} = -\frac{c_n}{n+2}, \quad n = 0, 1, 2, \ldots
\]

Then the even coefficients are given by \( c_2 = \frac{c_0}{2} \), \( c_4 = \frac{c_0}{4} \), \( c_6 = \frac{c_0}{6} \), \( c_8 = -\frac{c_0}{8 \cdot 4 \cdot 2} = -\frac{c_0}{2^4 n!} \), and in general, \( c_{2 n} = -\frac{c_0}{2^{2 n} n!} \). The odd coefficients are \( c_1 = \frac{c_1}{3} \), \( c_3 = -\frac{c_1}{5} \), \( c_5 = -\frac{c_1}{7} \), \( c_7 = \frac{c_1}{9} \), and in general, \( c_{2 n+1} = (-1)^{n+1} \frac{c_1}{3 \cdot 5 \cdot 7 \ldots (2 n+1)} = \frac{(-2)^n n! c_1}{(2 n+1)!} \). The solution is

\[
y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2 n} n!} x^{2 n} + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n n!}{(2 n+1)!} x^{2 n+1}.
\]

7. Let \( y(x) = \sum_{n=0}^{\infty} c_n x^n \). Then \( y'' = \sum_{n=0}^{\infty} (n-1) c_n x^{n-2} \), \( xy' = \sum_{n=0}^{\infty} n c_n x^n \) and

\[
(x^2 + 1) y'' = \sum_{n=0}^{\infty} n (n-1) c_n x^n + \sum_{n=0}^{\infty} (n+2) (n+1) c_{n+2} x^n. \text{ The differential equation becomes}
\]

\[
\sum_{n=0}^{\infty} [(n+2) (n+1) c_{n+2} + [n (n-1) + n - 1] c_n] x^n = 0. \text{ The recursion relation is } c_{n+2} = -\frac{(n-1) c_n}{n+2}.
\]
11. Assuming that \( y(x) = \sum_{n=0}^{\infty} c_n x^n \), we have \( xy = x \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+1} \),

\[
x^2 y' = x^2 \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} n c_n x^{n+1},
\]

\[
y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=1}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1} \quad \text{[replace } n \text{ with } n+3]\]

\[
= 2c_2 + \sum_{n=0}^{\infty} (n+3)(n+2) c_{n+3} x^{n+1},
\]

and the equation \( y'' + x^2 y' + xy = 0 \) becomes \( 2c_2 + \sum_{n=0}^{\infty} [(n+3)(n+2)c_{n+3} + nc_n + c_n] x^{n+1} = 0 \).

So \( c_2 = 0 \) and the recursion relation is \( c_{n+3} = \frac{-nc_n - c_n}{(n+3)(n+2)} = -\frac{(n+1)c_n}{(n+3)(n+2)}, \quad n = 0, 1, 2, \ldots \)

But \( c_0 = y(0) = c_2 = 0 \) and by the recursion relation, \( c_{3n} = c_{3n+2} = 0 \) for \( n = 0, 1, 2, \ldots \)

Also, \( c_1 = y'(0) = 1, \) so

\[
c_4 = -\frac{2c_1}{4 \cdot 3} = -\frac{2}{4 \cdot 3}, \quad c_7 = \frac{5c_4}{7 \cdot 6} = \frac{5 \cdot 2 \cdot 5}{7 \cdot 6 \cdot 4 \cdot 3} = \frac{(-1)^22^25^2}{7!}, \ldots,
\]

\[
c_{3n+1} = (-1)^n \frac{2^25^2 \cdots (3n-1)^2}{(3n+1)!}. \quad \text{Thus, the solution is}
\]

\[
y(x) = \sum_{n=0}^{\infty} c_n x^n = x + \sum_{n=1}^{\infty} \left[ (-1)^n \frac{2^25^2 \cdots (3n-1)^2}{(3n+1)!} \right]
\]