STRATEGY FOR TESTING SERIES

We now have several ways of testing a series for convergence or divergence; the problem is to decide which test to use on which series. In this respect testing series is similar to integrating functions. Again there are no hard and fast rules about which test to apply to a given series, but you may find the following advice of some use.

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its *form.*

1. If the series is of the form $\Sigma\, 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.
2. If the series has the form $\Sigma\, ar^{n-1}$ or $\Sigma\, ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. (The value of p should be chosen as in Section 8.3 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\Sigma\, a_n$ has some negative terms, then we can apply the Comparison Test to $\Sigma\, |a_n|$ and test for absolute convergence.
4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.
5. If the series is of the form $\Sigma\, (-1)^{n-1}b_n$ or $\Sigma\, (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.
6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.
7. If $a_n = f(n)$, where $\int_1^\infty f(x)\,dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

In the following examples we don't work out all the details but simply indicate which tests should be used.

EXAMPLE 1 $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

Since $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, we should use the Test for Divergence. ■

EXAMPLE 2 $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Since a_n is an algebraic function of n, we compare the given series with a p-series. The comparison series for the Limit Comparison Test is $\Sigma\, b_n$, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$ ■

EXAMPLE 3 $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

Since the integral $\int_1^\infty xe^{-x^2}\,dx$ is easily evaluated, we use the Integral Test. The Ratio Test also works. ■

EXAMPLE 4 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

Since the series is alternating, we use the Alternating Series Test. ■

EXAMPLE 5 $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$

Since the series involves $k!$, we use the Ratio Test. ■

EXAMPLE 6 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Since the series is closely related to the geometric series $\Sigma\, 1/3^n$, we use the Comparison Test. ■

EXERCISES

A Click here for answers.

S Click here for solutions.

1–34 ■ Test the series for convergence or divergence.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2+n}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$
4. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n-1}{n^2+n}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+n\cos^2 n}$
7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$
8. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$
9. $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-k}$
10. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$
11. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\ln n}$
12. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+25}$
13. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$
14. $\displaystyle\sum_{n=1}^{\infty} \sin n$
15. $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{2\cdot 5\cdot 8\cdot \cdots \cdot (3n+2)}$
16. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$
17. $\displaystyle\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$
18. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$
19. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$
20. $\displaystyle\sum_{k=1}^{\infty} \frac{k+5}{5^k}$
21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$
22. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$
23. $\displaystyle\sum_{n=1}^{\infty} \tan(1/n)$
24. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2+4n}$
25. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$
26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$
27. $\displaystyle\sum_{k=1}^{\infty} \frac{k\ln k}{(k+1)^3}$
28. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$
29. $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n\sqrt{n}}$
30. $\displaystyle\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$
31. $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{3^k+4^k}$
32. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$
33. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$
34. $\displaystyle\sum_{n=1}^{\infty} (\sqrt[n]{2}-1)$

ANSWERS

S Click here for solutions.

1. D **3.** C **5.** C **7.** D **9.** C **11.** C **13.** C
15. C **17.** D **19.** C **21.** C **23.** D **25.** C
27. C **29.** C **31.** D **33.** C

SOLUTIONS

1. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n^2-1}{n^2+1} = \lim_{n\to\infty} \dfrac{1-1/n^2}{1+1/n} = 1 \neq 0$, so the series $\sum_{n=1}^{\infty} \dfrac{n^2-1}{n^2+1}$ diverges by the Test for Divergence.

3. $\dfrac{1}{n^2+n} < \dfrac{1}{n^2}$ for all $n \geq 1$, so $\sum_{n=1}^{\infty} \dfrac{1}{n^2+n}$ converges by the Comparison Test with $\sum_{n=1}^{\infty} \dfrac{1}{n^2}$, a p-series that converges because $p = 2 > 1$.

5. $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{(-3)^{n+2}}{2^{3(n+1)}} \cdot \dfrac{2^{3n}}{(-3)^{n+1}}\right| = \lim_{n\to\infty} \left|\dfrac{-3 \cdot 2^{3n}}{2^{3n} \cdot 2^3}\right| = \lim_{n\to\infty} \dfrac{3}{2^3} = \dfrac{3}{8} < 1$, so the series $\sum_{n=1}^{\infty} \dfrac{(-3)^{n+1}}{2^{3n}}$ is absolutely convergent by the Ratio Test.

7. Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on $[2, \infty)$, so we can apply the Integral Test.

Since $\displaystyle\int \frac{1}{x\sqrt{\ln x}}\,dx \quad \begin{bmatrix} u = \ln x, \\ du = dx/x \end{bmatrix} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$, we find

$\displaystyle\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \int_2^t \frac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \left[2\sqrt{\ln x}\right]_2^t = \lim_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right) = \infty$. Since the integral diverges, the given series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges.

9. $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-k} = \sum_{k=1}^{\infty} \frac{k^2}{e^k}$. Using the Ratio Test, we get

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2}\right| = \lim_{k\to\infty} \left[\left(\frac{k+1}{k}\right)^2 \cdot \frac{1}{e}\right] = 1^2 \cdot \frac{1}{e} = \frac{1}{e} < 1,$$ so the series converges.

11. $b_n = \dfrac{1}{n\ln n} > 0$ for $n \geq 2$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the given series $\sum_{n=2}^{\infty} \dfrac{(-1)^{n+1}}{n\ln n}$ converges by the Alternating Series Test.

13. $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{3^{n+1}(n+1)^2}{(n+1)!} \cdot \dfrac{n!}{3^n n^2}\right| = \lim_{n\to\infty} \left[\dfrac{3(n+1)^2}{(n+1)n^2}\right] = 3\lim_{n\to\infty} \dfrac{n+1}{n^2} = 0 < 1$, so the series $\sum_{n=1}^{\infty} \dfrac{3^n n^2}{n!}$ converges by the Ratio Test.

15.
$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)!}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)[3(n+1)+2]} \cdot \frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)}{n!}\right| \\ &= \lim_{n\to\infty} \frac{n+1}{3n+5} = \frac{1}{3} < 1\end{aligned}$$

so the series $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{2\cdot 5\cdot 8\cdot\cdots\cdot(3n+2)}$ converges by the Ratio Test.

17. $\lim_{n\to\infty} 2^{1/n} = 2^0 = 1$, so $\lim_{n\to\infty} (-1)^n 2^{1/n}$ does not exist and the series $\sum_{n=1}^{\infty} (-1)^n 2^{1/n}$ diverges by the Test for Divergence.

19. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$. Then $f'(x) = \dfrac{2-\ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\dfrac{\ln n}{\sqrt{n}}$ is decreasing for $n > e^2$.

By l'Hospital's Rule, $\lim_{n\to\infty} \dfrac{\ln n}{\sqrt{n}} = \lim_{n\to\infty} \dfrac{1/n}{1/(2\sqrt{n})} = \lim_{n\to\infty} \dfrac{2}{\sqrt{n}} = 0$, so the series $\sum_{n=1}^{\infty} (-1)^n \dfrac{\ln n}{\sqrt{n}}$ converges by the Alternating Series Test.

21. $\sum_{n=1}^{\infty} \dfrac{(-2)^{2n}}{n^n} = \sum_{n=1}^{\infty} \left(\dfrac{4}{n}\right)^n$. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \dfrac{4}{n} = 0 < 1$, so the given series is absolutely convergent by the Root Test.

23. Using the Limit Comparison Test with $a_n = \tan\left(\dfrac{1}{n}\right)$ and $b_n = \dfrac{1}{n}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\tan(1/n)}{1/n} = \lim_{x\to\infty} \frac{\tan(1/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\sec^2(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} \sec^2(1/x) = 1^2 = 1 > 0.$$

Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ is also divergent.

25. Use the Ratio Test. $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{(n+1)!}{e^{(n+1)^2}} \cdot \dfrac{e^{n^2}}{n!}\right| = \lim_{n\to\infty} \dfrac{(n+1)n! \cdot e^{n^2}}{e^{n^2+2n+1} n!} = \lim_{n\to\infty} \dfrac{n+1}{e^{2n+1}} = 0 < 1$, so $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$ converges.

27. $\displaystyle\int_2^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t$ (using integration by parts) $\overset{\text{H}}{=} 1$. So $\sum_{n=1}^{\infty} \dfrac{\ln n}{n^2}$ converges by the Integral Test, and since $\dfrac{k \ln k}{(k+1)^3} < \dfrac{k \ln k}{k^3} = \dfrac{\ln k}{k^2}$, the given series $\sum_{k=1}^{\infty} \dfrac{k \ln k}{(k+1)^3}$ converges by the Comparison Test.

29. $0 < \dfrac{\tan^{-1} n}{n^{3/2}} < \dfrac{\pi/2}{n^{3/2}}$. $\sum_{n=1}^{\infty} \dfrac{\pi/2}{n^{3/2}} = \dfrac{\pi}{2} \sum_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ which is a convergent p-series ($p = \frac{3}{2} > 1$), so $\sum_{n=1}^{\infty} \dfrac{\tan^{-1} n}{n^{3/2}}$ converges by the Comparison Test.

31. $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \dfrac{5^k}{3^k + 4^k} =$ [divide by 4^k] $\lim_{k\to\infty} \dfrac{(5/4)^k}{(3/4)^k + 1} = \infty$ since $\lim_{k\to\infty} \left(\dfrac{3}{4}\right)^k = 0$ and $\lim_{k\to\infty} \left(\dfrac{5}{4}\right)^k = \infty$. Thus, $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$ diverges by the Test for Divergence.

33. Let $a_n = \dfrac{\sin(1/n)}{\sqrt{n}}$ and $b_n = \dfrac{1}{n\sqrt{n}}$. Then $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{\sin(1/n)}{1/n} = 1 > 0$, so $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ converges by limit comparison with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $(p = 3/2 > 1)$.