There are two situations in which it is impossible to find the exact value of a definite integral.

The first situation arises from the fact that in order to evaluate \( \int_a^b f(x) \, dx \) using the Fundamental Theorem of Calculus we need to know an antiderivative of \( f \). Sometimes, however, it is difficult, or even impossible, to find an antiderivative (see Section 5.7). For example, it is impossible to evaluate the following integrals exactly:

\[
\int_0^1 e^{x^2} \, dx
\]

\[
\int_{-1}^1 \sqrt{1 + x^3} \, dx
\]

The second situation arises when the function is determined from a scientific experiment through instrument readings or collected data. There may be no formula for the function (see Example 5).

In both cases we need to find approximate values of definite integrals. We already know one such method. Recall that the definite integral is defined as a limit of Riemann sums, so any Riemann sum could be used as an approximation to the integral: If we divide \([a, b]\) into \( n \) subintervals of equal length \( \Delta x = \frac{b-a}{n} \), then we have

\[
\int_a^b f(x) \, dx = \sum_{i=1}^{n} f(x_i^*) \Delta x
\]

where \( x_i^* \) is any point in the \( i \)th subinterval \([x_{i-1}, x_i]\). If \( x_i^* \) is chosen to be the left endpoint of the interval, then \( x_i^* = x_{i-1} \) and we have

\[
\int_a^b f(x) \, dx = L_n = \sum_{i=1}^{n} f(x_{i-1}) \Delta x
\]

If \( f(x) \geq 0 \), then the integral represents an area and (1) represents an approximation of this area by the rectangles shown in Figure 1(a). If we choose \( x_i^* \) to be the right endpoint, then \( x_i^* = x_i \), and we have

\[
\int_a^b f(x) \, dx = R_n = \sum_{i=1}^{n} f(x_i) \Delta x
\]

[See Figure 1(b).] The approximations \( L_n \) and \( R_n \) defined by Equations 1 and 2 are called the left endpoint approximation and right endpoint approximation, respectively.

We have also considered the case where \( x_i^* \) is chosen to be the midpoint \( x_i \) of the subinterval \([x_{i-1}, x_i]\). Figure 1(c) shows the midpoint approximation \( M_n \), which appears to be better than either \( L_n \) or \( R_n \).

<table>
<thead>
<tr>
<th>FIGURE 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Left endpoint approximation</td>
</tr>
<tr>
<td>(b) Right endpoint approximation</td>
</tr>
<tr>
<td>(c) Midpoint approximation</td>
</tr>
</tbody>
</table>
Another approximation, called the Trapezoidal Rule, results from averaging the approximations in Equations 1 and 2:

\[
\int_a^b f(x) \, dx \approx \frac{1}{2} \left[ \sum_{i=1}^{n} f(x_{i-1}) \Delta x + \sum_{i=1}^{n} f(x_i) \Delta x \right] = \frac{\Delta x}{2} \left[ \sum_{i=1}^{n} (f(x_{i-1}) + f(x_i)) \right]
\]

\[
= \frac{\Delta x}{2} \left[ (f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \cdots + (f(x_{n-1}) + f(x_n)) \right]
\]

\[
= \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]
\]

**Trapezoidal Rule**

\[
\int_a^b f(x) \, dx \approx T_n = \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]
\]

where \( \Delta x = (b - a)/n \) and \( x_i = a + i \Delta x \).

The reason for the name Trapezoidal Rule can be seen from Figure 2, which illustrates the case with \( f(x) \geq 0 \) and \( n = 4 \). The area of the trapezoid that lies above the \( i \)th subinterval is

\[
\Delta x \left( \frac{f(x_{i-1}) + f(x_i)}{2} \right) = \frac{\Delta x}{2} [f(x_{i-1}) + f(x_i)]
\]

and if we add the areas of all these trapezoids, we get the right side of the Trapezoidal Rule.

**EXAMPLE 1** Use (a) the Trapezoidal Rule and (b) the Midpoint Rule with \( n = 5 \) to approximate the integral \( \int_1^2 (1/x) \, dx \).

**SOLUTION**

(a) With \( n = 5 \), \( a = 1 \), and \( b = 2 \), we have \( \Delta x = (2 - 1)/5 = 0.2 \), and so the Trapezoidal Rule gives

\[
\int_1^2 \frac{1}{x} \, dx \approx T_5 = \frac{0.2}{2} \left[ f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2) \right]
\]

\[
= 0.1 \left( \frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2} \right)
\]

\[
= 0.695635
\]

This approximation is illustrated in Figure 3.

(b) The midpoints of the five subintervals are 1.1, 1.3, 1.5, 1.7, and 1.9, so the Midpoint Rule gives

\[
\int_1^2 \frac{1}{x} \, dx = \Delta x \left[ f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9) \right]
\]

\[
= \frac{1}{5} \left( \frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9} \right)
\]

\[
= 0.691908
\]

This approximation is illustrated in Figure 4.
In Example 1 we deliberately chose an integral whose value can be computed explicitly so that we can see how accurate the Trapezoidal and Midpoint Rules are. By the Fundamental Theorem of Calculus,

\[ \int_1^2 \frac{1}{x} \, dx = \ln x \bigg|_1^2 = \ln 2 = 0.693147 \ldots \]

The error in using an approximation is defined to be the amount that needs to be added to the approximation to make it exact. From the values in Example 1 we see that the errors in the Trapezoidal and Midpoint Rule approximations for \( n = 5 \) are

\[ E_T = -0.002488 \quad \text{and} \quad E_M = 0.001239 \]

In general, we have

\[ E_T = \int_a^b f(x) \, dx - T_n \quad \text{and} \quad E_M = \int_a^b f(x) \, dx - M_n \]

The following tables show the results of calculations similar to those in Example 1, but for \( n = 5, 10, \) and \( 20 \) and for the left and right endpoint approximations as well as the Trapezoidal and Midpoint Rules.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( L_n )</th>
<th>( R_n )</th>
<th>( T_n )</th>
<th>( M_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.745635</td>
<td>0.645635</td>
<td>0.695635</td>
<td>0.691908</td>
</tr>
<tr>
<td>10</td>
<td>0.718771</td>
<td>0.668771</td>
<td>0.693771</td>
<td>0.692835</td>
</tr>
<tr>
<td>20</td>
<td>0.705803</td>
<td>0.680803</td>
<td>0.693303</td>
<td>0.693069</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n )</th>
<th>( E_L )</th>
<th>( E_R )</th>
<th>( E_T )</th>
<th>( E_M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>-0.052488</td>
<td>0.047512</td>
<td>-0.002488</td>
<td>0.001239</td>
</tr>
<tr>
<td>10</td>
<td>-0.025624</td>
<td>0.024376</td>
<td>-0.000624</td>
<td>0.000312</td>
</tr>
<tr>
<td>20</td>
<td>-0.012656</td>
<td>0.012344</td>
<td>-0.000156</td>
<td>0.000078</td>
</tr>
</tbody>
</table>

We can make several observations from these tables:

1. In all of the methods we get more accurate approximations when we increase the value of \( n \). (But very large values of \( n \) result in so many arithmetic operations that we have to beware of accumulated round-off error.)

2. The errors in the left and right endpoint approximations are opposite in sign and appear to decrease by a factor of about 2 when we double the value of \( n \).

3. The Trapezoidal and Midpoint Rules are much more accurate than the endpoint approximations.

4. The errors in the Trapezoidal and Midpoint Rules are opposite in sign and appear to decrease by a factor of about 4 when we double the value of \( n \).

5. The size of the error in the Midpoint Rule is about half the size of the error in the Trapezoidal Rule.

Figure 5 shows why we can usually expect the Midpoint Rule to be more accurate than the Trapezoidal Rule. The area of a typical rectangle in the Midpoint Rule is the same as the area of the trapezoid \( ABCD \) whose upper side is tangent to the graph at \( P \). The area of this trapezoid is closer to the area under the graph than is the area of the trapezoid \( AQRD \) used in the Trapezoidal Rule. [The midpoint error (shaded red) is smaller than the trapezoidal error (shaded blue).]
These observations are corroborated in the following error estimates, which are proved in books on numerical analysis. Notice that Observation 4 corresponds to the $n^2$ in each denominator because $(2n)^2 = 4n^2$. The fact that the estimates depend on the size of the second derivative is not surprising if you look at Figure 5, because $f''(x)$ measures how much the graph is curved. [Recall that $f''(x)$ measures how fast the slope of $y = f(x)$ changes.]

### Error Bounds

Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If $E_T$ and $E_M$ are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leq \frac{K(b - a)^3}{12n^2} \quad \text{and} \quad |E_M| \leq \frac{K(b - a)^3}{24n^2}$$

Let’s apply this error estimate to the Trapezoidal Rule approximation in Example 1. If $f(x) = 1/x$, then $f'(x) = -1/x^2$ and $f''(x) = 2/x^3$. Because $1 \leq x \leq 2$, we have $1/x \leq 1$, so

$$|f''(x)| = \left| \frac{2}{x^3} \right| \leq \frac{2}{1^3} = 2$$

Therefore, taking $K = 2, a = 1, b = 2,$ and $n = 5$ in the error estimate (3), we see that

$$|E_T| \leq \frac{2(2 - 1)^3}{12(5)^2} = \frac{1}{150} \approx 0.006667$$

Comparing this error estimate of 0.006667 with the actual error of about 0.002488, we see that it can happen that the actual error is substantially less than the upper bound for the error given by (3).

### Example 2

How large should we take $n$ in order to guarantee that the Trapezoidal and Midpoint Rule approximations for $\int_1^2 (1/x) \, dx$ are accurate to within 0.0001?

**Solution** We saw in the preceding calculation that $|f''(x)| \leq 2$ for $1 \leq x \leq 2$, so we can take $K = 2, a = 1, b = 2$ in (3). Accuracy to within 0.0001 means that the size of the error should be less than 0.0001. Therefore we choose $n$ so that

$$\frac{2(1)^3}{12n^2} < 0.0001$$

Solving the inequality for $n$, we get

$$n^2 > \frac{2}{12(0.0001)}$$

or

$$n > \frac{1}{\sqrt{0.0006}} \approx 40.8$$

Thus $n = 41$ will ensure the desired accuracy.

For the same accuracy with the Midpoint Rule we choose $n$ so that

$$\frac{2(1)^3}{24n^2} < 0.0001 \quad \text{and so} \quad n > \frac{1}{\sqrt{0.0012}} \approx 29$$
EXAMPLE 3
(a) Use the Midpoint Rule with \( n = 10 \) to approximate the integral \( \int_0^1 e^{x^2} \, dx \).
(b) Give an upper bound for the error involved in this approximation.

SOLUTION
(a) Since \( a = 0, b = 1, \) and \( n = 10 \), the Midpoint Rule gives
\[
\int_0^1 e^{x^2} \, dx \approx \Delta x \left[ f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95) \right]
\]
\[
= 0.1 \left[ e^{0.0025} + e^{0.0225} + e^{0.0625} + e^{0.1225} + e^{0.2025} + e^{0.3025} + e^{0.4225} + e^{0.5625} + e^{0.7225} + e^{0.9025} \right]
\]
\[
= 1.460393
\]

Figure 6 illustrates this approximation.

(b) Since \( f(x) = e^{x^2} \), we have \( f'(x) = 2xe^{x^2} \) and \( f''(x) = (2 + 4x^2)e^{x^2} \). Also, since
\( 0 \leq x \leq 1 \), we have \( x^2 \leq 1 \) and so
\[
0 \leq f''(x) = (2 + 4x^2)e^{x^2} \leq 6e
\]

Taking \( K = 6e, \) \( a = 0, b = 1, \) and \( n = 10 \) in the error estimate (3), we see that an upper bound for the error is
\[
\frac{6e(1)^3}{24(10)^2} = \frac{e}{400} \approx 0.007
\]

Simpson’s Rule
Another rule for approximate integration results from using parabolas instead of straight line segments to approximate a curve. As before, we divide \([a, b]\) into \( n \) subintervals of equal length \( h = \Delta x = (b - a)/n \), but this time we assume that \( n \) is an even number. Then on each consecutive pair of intervals we approximate the curve \( y = f(x) \geq 0 \) by a parabola as shown in Figure 7. If \( y_i = f(x_i) \), then \( P_i(x, y) \) is the point on the curve lying above \( x_i \). A typical parabola passes through three consecutive points \( P_i, P_{i+1}, \) and \( P_{i+2} \).

To simplify our calculations, we first consider the case where \( x_0 = -h, x_1 = 0, \) and \( x_2 = h \). (See Figure 8.) We know that the equation of the parabola through \( P_0, P_1, \) and \( P_2 \) is of the form \( y = Ax^2 + Bx + C \) and so the area under the parabola from \( x = -h \)
Here we have used Theorem 5.4.6.
Notice that $Ax^2 + C$ is even and $Bx$ is odd.

\[
\int_{-h}^{h} (Ax^2 + Bx + C) \, dx = 2 \int_{0}^{h} (Ax^2 + C) \, dx = 2 \left[ \frac{Ax^3}{3} + Cx \right]_{0}^{h} = 2 \left( \frac{Ah^3}{3} + Ch \right) = \frac{h}{3} (2Ah^2 + 6C)
\]

But, since the parabola passes through $P_0(-h, y_0)$, $P_1(0, y_1)$, and $P_2(h, y_2)$, we have
\[
y_0 = A(-h)^2 + B(-h) + C = Ah^2 - Bh + C
\]
\[
y_1 = C
\]
\[
y_2 = Ah^2 + Bh + C
\]
and therefore
\[
y_0 + 4y_1 + y_2 = 2Ah^2 + 6C
\]
Thus we can rewrite the area under the parabola as
\[
\frac{h}{3} (y_0 + 4y_1 + y_2)
\]
Now by shifting this parabola horizontally we do not change the area under it. This means that the area under the parabola through $P_0$, $P_1$, and $P_2$ from $x = x_0$ to $x = x_2$ in Figure 7 is still
\[
\frac{h}{3} (y_0 + 4y_1 + y_2)
\]
Similarly, the area under the parabola through $P_2$, $P_3$, and $P_4$ from $x = x_2$ to $x = x_4$ is
\[
\frac{h}{3} (y_2 + 4y_3 + y_4)
\]
If we compute the areas under all the parabolas in this manner and add the results, we get
\[
\int_{a}^{b} f(x) \, dx \approx S_n = \frac{\Delta x}{3} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]
\]
Although we have derived this approximation for the case in which $f(x) \geq 0$, it is a reasonable approximation for any continuous function $f$ and is called Simpson’s Rule after the English mathematician Thomas Simpson (1710–1761). Note the pattern of coefficients: 1, 4, 2, 4, 2, 4, 2, ..., 4, 2, 4, 1.

**Simpson's Rule**
\[
\int_{a}^{b} f(x) \, dx \approx S_n = \frac{\Delta x}{3} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]
\]
where $n$ is even and $\Delta x = (b-a)/n$. 

**Simpson**
Thomas Simpson was a weaver who taught himself mathematics and went on to become one of the best English mathematicians of the 18th century. What we call Simpson’s Rule was actually known to Cavalieri and Gregory in the 17th century, but Simpson popularized it in his book *Mathematical Dissertations* (1743).
EXAMPLE 4 Use Simpson’s Rule with \( n = 10 \) to approximate \( \int_1^2 \frac{1}{x} \, dx \).

SOLUTION Putting \( f(x) = \frac{1}{x}, n = 10 \), and \( \Delta x = 0.1 \) in Simpson’s Rule, we obtain

\[
\int_1^2 \frac{1}{x} \, dx \approx S_{10} \\
= \frac{\Delta x}{3} \left[ f(1) + 4 f(1.1) + 2 f(1.2) + 4 f(1.3) + \cdots + 2 f(1.8) + 4 f(1.9) + f(2) \right] \\
= \frac{0.1}{3} \left( \frac{1}{1} + \frac{4}{1.1} + \frac{2}{1.2} + \frac{4}{1.3} + \frac{2}{1.4} + \frac{4}{1.5} + \frac{2}{1.6} + \frac{4}{1.7} + \frac{2}{1.8} + \frac{4}{1.9} + \frac{1}{2} \right) \\
\approx 0.693150
\]

Notice that, in Example 4, Simpson’s Rule gives us a much better approximation \( S_{10} \approx 0.693150 \) to the true value of the integral \( \ln 2 \approx 0.693147 \ldots \) than does the Trapezoidal Rule \( T_{10} \approx 0.693771 \) or the Midpoint Rule \( M_{10} \approx 0.692835 \). It turns out (see Exercise 50) that the approximations in Simpson’s Rule are weighted averages of those in the Trapezoidal and Midpoint Rules:

\[
S_{2n} = \frac{1}{2}T_n + \frac{2}{3}M_n
\]

(Recall that \( E_T \) and \( E_M \) usually have opposite signs and \( |E_M| \) is about half the size of \( |E_T| \).)

In many applications of calculus we need to evaluate an integral even if no explicit formula is known for \( y \) as a function of \( x \). A function may be given graphically or as a table of values of collected data. If there is evidence that the values are not changing rapidly, then the Trapezoidal Rule or Simpson’s Rule can still be used to find an approximate value for \( \int_a^b y \, dx \), the integral of \( y \) with respect to \( x \).

EXAMPLE 5 Figure 9 shows data traffic on the link from the United States to SWITCH, the Swiss academic and research network, on February 10, 1998. \( D(t) \) is the data throughput, measured in megabits per second (Mb/s). Use Simpson’s Rule to estimate the total amount of data transmitted on the link from midnight to noon on that day.

SOLUTION Because we want the units to be consistent and \( D(t) \) is measured in megabits per second, we convert the units for \( t \) from hours to seconds. If we let \( A(t) \) be the amount of data (in megabits) transmitted by time \( t \), where \( t \) is measured in seconds, then \( A'(t) = D(t) \). So, by the Net Change Theorem (see Section 5.4), the total amount
of data transmitted by noon (when \( t = 12 \times 60^2 = 43,200 \)) is

\[
A(43,200) = \int_0^{43,200} D(t) \, dt
\]

We estimate the values of \( D(t) \) at hourly intervals from the graph and compile them in the table.

<table>
<thead>
<tr>
<th>( t ) (hours)</th>
<th>( t ) (seconds)</th>
<th>( D(t) )</th>
<th>( t ) (hours)</th>
<th>( t ) (seconds)</th>
<th>( D(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>3.2</td>
<td>7</td>
<td>25,200</td>
<td>1.3</td>
</tr>
<tr>
<td>1</td>
<td>3,600</td>
<td>2.7</td>
<td>8</td>
<td>28,800</td>
<td>2.8</td>
</tr>
<tr>
<td>2</td>
<td>7,200</td>
<td>1.9</td>
<td>9</td>
<td>32,400</td>
<td>5.7</td>
</tr>
<tr>
<td>3</td>
<td>10,800</td>
<td>1.7</td>
<td>10</td>
<td>36,000</td>
<td>7.1</td>
</tr>
<tr>
<td>4</td>
<td>14,400</td>
<td>1.3</td>
<td>11</td>
<td>39,600</td>
<td>7.7</td>
</tr>
<tr>
<td>5</td>
<td>18,000</td>
<td>1.0</td>
<td>12</td>
<td>43,200</td>
<td>7.9</td>
</tr>
<tr>
<td>6</td>
<td>21,600</td>
<td>1.1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Then we use Simpson’s Rule with \( n = 12 \) and \( \Delta t = 3600 \) to estimate the integral:

\[
\int_0^{43,200} A(t) \, dt \approx \frac{\Delta t}{3} \left[ D(0) + 4D(3600) + 2D(7200) + \cdots + 4D(39,600) + D(43,200) \right]
\]

\[
\approx \frac{3600}{3} \left[ 3.2 + 4(2.7) + 2(1.9) + 4(1.7) + 2(1.3) + 4(1.0) \right.
\]

\[
+ 2(1.1) + 4(1.3) + 2(2.8) + 4(5.7) + 2(7.1) + 4(7.7) + 7.9 \right]
\]

\[
= 143,880
\]

Thus the total amount of data transmitted from midnight to noon is about 144,000 megabits, or 144 gigabits.

The table in the margin shows how Simpson’s Rule compares with the Midpoint Rule for the integral \( \int_0^1 x \, dx \), whose value is about 0.69314718. The second table shows how the error \( E_S \) in Simpson’s Rule decreases by a factor of about 16 when \( n \) is doubled. (In Exercises 27 and 28 you are asked to verify this for two additional integrals.) That is consistent with the appearance of \( n^4 \) in the denominator of the following error estimate for Simpson’s Rule. It is similar to the estimates given in (3) for the Trapezoidal and Midpoint Rules, but it uses the fourth derivative of \( f \).

\[\text{Error Bound for Simpson’s Rule} \quad \text{Suppose that } \left| f^{(4)}(x) \right| \leq K \text{ for } a \leq x \leq b. \]

If \( E_S \) is the error involved in using Simpson’s Rule, then

\[
|E_S| \leq \frac{K(b-a)^5}{180n^4}
\]

**EXAMPLE 6** How large should we take \( n \) in order to guarantee that the Simpson’s Rule approximation for \( \int_1^2 (1/x) \, dx \) is accurate to within 0.0001?
Many calculators and computer algebra systems have a built-in algorithm that computes an approximation of a definite integral. Some of these machines use Simpson’s Rule; others use more sophisticated techniques such as adaptive numerical integration. This means that if a function fluctuates much more on a certain part of the interval than it does elsewhere, then that part gets divided into more subintervals. This strategy reduces the number of calculations required to achieve a prescribed accuracy.

**Example 7**

(a) Use Simpson’s Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2} \, dx$.

(b) Estimate the error involved in this approximation.

**Solution**

(a) If $n = 10$, then $\Delta x = 0.1$ and Simpson’s Rule gives

\[
\int_0^1 e^{x^2} \, dx \approx \frac{\Delta x}{3} \left[ f(0) + 4f(0.1) + 2f(0.2) + \cdots + 2f(0.8) + 4f(0.9) + f(1) \right]
\]

\[
= \frac{0.1}{3} \left[ e^0 + 4e^{0.01} + 2e^{0.04} + 4e^{0.09} + 2e^{0.16} + 4e^{0.25} + 2e^{0.36} + 4e^{0.49} + 2e^{0.64} + 4e^{0.81} + e^1 \right]
\]

\[
= 1.462681
\]

(b) The fourth derivative of $f(x) = e^{x^2}$ is

\[
f^{(4)}(x) = (12 + 48x^2 + 16x^4)e^{x^2}
\]

and so, since $0 \leq x \leq 1$, we have

\[
0 \leq f^{(4)}(x) \leq (12 + 48 + 16)e^1 = 76e
\]

Therefore, putting $K = 76e$, $a = 0$, $b = 1$, and $n = 10$ in (4), we see that the error is at most

\[
\frac{76e(1)^5}{180(10)^4} \approx 0.000115
\]

(Compare this with Example 3.) Thus, correct to three decimal places, we have

\[
\int_0^1 e^{x^2} \, dx \approx 1.463
\]
1. Let \( I = \int_{a}^{b} f(x) \, dx \), where \( f \) is the function whose graph is shown.
   (a) Use the graph to find \( L_2, R_3, \) and \( M_2. \)
   (b) Are these underestimates or overestimates of \( I? \)
   (c) Use the graph to find \( T_3. \) How does it compare with \( I? \)
   (d) For any value of \( n, \) list the numbers \( L_n, R_n, M_n, T_n, \) and \( I \)
   in increasing order.

2. The left, right, Trapezoidal, and Midpoint Rule approximations were used to estimate \( \int_{a}^{b} f(x) \, dx \), where \( f \) is the function whose graph is shown. The estimates were 0.7811, 0.8675, 0.8632, and 0.9540, and the same number of sub-intervals were used in each case.
   (a) Which rule produced which estimate?
   (b) Between which two approximations does the true value of \( \int_{a}^{b} f(x) \, dx \) lie?

3. Estimate \( \int_{1}^{3} \cos(x^2) \, dx \) using (a) the Trapezoidal Rule and
   (b) the Midpoint Rule, each with \( n = 4. \) From a graph of the integrand, decide whether your answers are underestimates or overestimates. What can you conclude about the true value of the integral?

4. Draw the graph of \( f(x) = \sin\left(\frac{1}{x^3}\right) \) in the viewing rectangle \([0, 1] \) by \([0, 0.5] \) and let \( I = \int_{0}^{1} f(x) \, dx \).
   (a) Use the graph to decide whether \( L_2, R_2, M_2, \) and \( T_2 \)
   underestimate or overestimate \( I \).
   (b) For any value of \( n, \) list the numbers \( L_n, R_n, M_n, T_n, \) and \( I \)
   in increasing order.
   (c) Compute \( L_1, R_1, M_1, \) and \( T_1. \) From the graph, which do you think gives the best estimate of \( I? \)

5–6 Use (a) the Midpoint Rule and (b) Simpson’s Rule to approximate the given integral with the specified value of \( n. \) (Round your answers to six decimal places.) Compare your results to the actual value to determine the error in each approximation.

<table>
<thead>
<tr>
<th>( \int_{0}^{1} x \cos x , dx, \ n = 4 )</th>
<th>( \int_{0}^{1} x \sin x , dx, \ n = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.24000000</td>
<td>5.24000000</td>
</tr>
</tbody>
</table>

7–18 Use (a) the Trapezoidal Rule, (b) the Midpoint Rule, and
(c) Simpson’s Rule to approximate the given integral with the specified value of \( n. \) (Round your answers to six decimal places.) Compare your results to the actual value to determine the error in each approximation.

<table>
<thead>
<tr>
<th>( \int_{0}^{1} \sqrt{x^3 - 1} , dx, \ n = 10 )</th>
<th>( \int_{0}^{1} \frac{1}{x + 1} , dx, \ n = 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.00000000</td>
<td>1.50000000</td>
</tr>
</tbody>
</table>

19. (a) Find the approximations \( T_n \) and \( M_n \) for the integral \( \int_{0}^{1} \cos(x^2) \, dx \).
   (b) Estimate the errors in the approximations of part (a).
   (c) How large do we have to choose \( n \) so that the approximations \( T_n \) and \( M_n \)
   to the integral in part (a) are accurate to within 0.0001?

20. (a) Find the approximations \( T_{10} \) and \( M_{10} \) for \( \int_{0}^{1} e^{1/x} \, dx. \)
   (b) Estimate the errors in the approximations of part (a).
   (c) How large do we have to choose \( n \) so that the approximations \( T_n \) and \( M_n \)
   to the integral in part (a) are accurate to within 0.0001?

21. (a) Find the approximations \( T_{10}, M_{10}, \) and \( S_{10} \) for \( \int_{0}^{1} \sin(x) \, dx \)
   and the corresponding errors \( E_T, E_M, \) and \( E_S. \)
   (b) Compare the actual errors in part (a) with the error estimates given by (3) and (4).
   (c) How large do we have to choose \( n \) so that the approximations \( T_n, \) \( M_n, \) and \( S_n \)
   to the integral in part (a) are accurate to within 0.00001?

22. How large should \( n \) be to guarantee that the Simpson’s Rule approximation to \( \int_{0}^{1} e^{x} \, dx \) is accurate to within 0.00001?

23. The trouble with the error estimates is that it is often very difficult to compute four derivatives and obtain a good upper bound \( K \) for \( |f^{(4)}(x)| \) by hand. But computer algebra systems have no problem computing \( f^{(4)} \) and graphing it, so we can easily find a value for \( K \) from a machine graph.
   This exercise deals with approximations to the integral \( I = \int_{0}^{1} f(x) \, dx, \) where \( f(x) = e^{\cos x}. \)
   (a) Use a graph to get a good upper bound for \( |f^{(4)}(x)|. \)
   (b) Use \( M_{10} \) to approximate \( I. \)
24. Repeat Exercise 23 for the integral \( \int_{1}^{4} \sqrt{4 - x^3} \, dx \).

25–26 Find the approximations \( L_n, R_n, T_n, \) and \( M_n \) for \( n = 5, 10, \) and 20. Then compute the corresponding errors \( E_L, E_R, E_T, \) and \( E_M \). (Round your answers to six decimal places. You may wish to use the sum command on a computer algebra system.) What observations can you make? In particular, what happens to the errors when \( n \) is doubled?

25. \( \int_{0}^{1} xe^x \, dx \)

26. \( \int_{1}^{3} \frac{1}{x^2} \, dx \)

27–28 Find the approximations \( T_n, M_n, \) and \( S_n \) for \( n = 6 \) and 12. Then compute the corresponding errors \( E_T, E_M, \) and \( E_S \). (Round your answers to six decimal places. You may wish to use the sum command on a computer algebra system.) What observations can you make? In particular, what happens to the errors when \( n \) is doubled?

27. \( \int_{0}^{1} x^4 \, dx \)

28. \( \int_{1}^{3} \frac{1}{x^3} \, dx \)

29. Estimate the area under the graph in the figure by using (a) the Trapezoidal Rule, (b) the Midpoint Rule, and (c) Simpson’s Rule, each with \( n = 6 \).

30. The widths (in meters) of a kidney-shaped swimming pool were measured at 2-meter intervals as indicated in the figure. Use Simpson’s Rule to estimate the area of the pool.

31. (a) Use the Midpoint Rule and the given data to estimate the value of the integral \( \int_{1}^{2} f(x) \, dx \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>2.4</td>
<td>3.5</td>
<td>4.0</td>
</tr>
<tr>
<td>1.5</td>
<td>2.9</td>
<td>4.0</td>
<td>4.1</td>
</tr>
<tr>
<td>2.0</td>
<td>3.3</td>
<td>4.5</td>
<td>3.9</td>
</tr>
<tr>
<td>2.5</td>
<td>3.6</td>
<td>5.0</td>
<td>3.5</td>
</tr>
<tr>
<td>3.0</td>
<td>3.8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) If it is known that \(-2 \leq f''(x) \leq 3\) for all \( x \), estimate the error involved in the approximation in part (a).

32. (a) A table of values of a function \( g(x) \) is given. Use Simpson’s Rule to estimate \( \int_{0}^{1} g(x) \, dx \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( g(x) )</th>
<th>( x )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>12.1</td>
<td>1.0</td>
<td>12.2</td>
</tr>
<tr>
<td>0.2</td>
<td>11.6</td>
<td>1.2</td>
<td>12.6</td>
</tr>
<tr>
<td>0.4</td>
<td>11.3</td>
<td>1.4</td>
<td>13.0</td>
</tr>
<tr>
<td>0.6</td>
<td>11.1</td>
<td>1.6</td>
<td>13.2</td>
</tr>
<tr>
<td>0.8</td>
<td>11.7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) If \(-5 \leq g''(x) \leq 2\) for \( 0 \leq x \leq 1.6 \), estimate the error involved in the approximation in part (a).

33. A graph of the temperature in Boston on August 11, 2013, is shown. Use Simpson’s Rule with \( n = 12 \) to estimate the average temperature on that day.

34. A radar gun was used to record the speed of a runner during the first 5 seconds of a race (see the table). Use Simpson’s Rule to estimate the distance the runner covered during those 5 seconds.

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>( v ) (m/s)</th>
<th>( t ) (s)</th>
<th>( v ) (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>3.0</td>
<td>10.51</td>
</tr>
<tr>
<td>0.5</td>
<td>4.67</td>
<td>3.5</td>
<td>10.67</td>
</tr>
<tr>
<td>1.0</td>
<td>7.34</td>
<td>4.0</td>
<td>10.76</td>
</tr>
<tr>
<td>1.5</td>
<td>8.86</td>
<td>4.5</td>
<td>10.81</td>
</tr>
<tr>
<td>2.0</td>
<td>9.73</td>
<td>5.0</td>
<td>10.81</td>
</tr>
<tr>
<td>2.5</td>
<td>10.22</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
35. The graph of the acceleration $a(t)$ of a car measured in ft/s² is shown. Use Simpson’s Rule to estimate the increase in the velocity of the car during the 6-second time interval.

36. Water leaked from a tank at a rate of $r(t)$ liters per hour, where the graph of $r$ is as shown. Use Simpson’s Rule to estimate the total amount of water that leaked out during the first 6 hours.

37. The table (supplied by San Diego Gas and Electric) gives the power consumption $P$ in megawatts in San Diego County from midnight to 6:00 am on a day in December. Use Simpson’s Rule to estimate the energy used during that time period. (Use the fact that power is the derivative of energy.)

<table>
<thead>
<tr>
<th>$t$</th>
<th>$P$</th>
<th>$t$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0:00</td>
<td>1814</td>
<td>3:30</td>
<td>1611</td>
</tr>
<tr>
<td>0:30</td>
<td>1735</td>
<td>4:00</td>
<td>1621</td>
</tr>
<tr>
<td>1:00</td>
<td>1686</td>
<td>4:30</td>
<td>1666</td>
</tr>
<tr>
<td>1:30</td>
<td>1646</td>
<td>5:00</td>
<td>1745</td>
</tr>
<tr>
<td>2:00</td>
<td>1637</td>
<td>5:30</td>
<td>1886</td>
</tr>
<tr>
<td>2:30</td>
<td>1609</td>
<td>6:00</td>
<td>2052</td>
</tr>
<tr>
<td>3:00</td>
<td>1604</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

38. Shown is the graph of traffic on an Internet service provider’s T1 data line from midnight to 8:00 am. $D$ is the data throughput, measured in megabits per second. Use Simpson’s Rule to estimate the total amount of data transmitted during that time period.

39. Use Simpson’s Rule when $n = 8$ to estimate the volume of the solid obtained by rotating the region shown in the figure about (a) the $x$-axis and (b) the $y$-axis.

40. The table shows values of a force function $f(x)$, where $x$ is measured in meters and $f(x)$ in newtons. Use Simpson’s Rule to estimate the work done by the force in moving an object a distance of 18 m.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9.8</td>
</tr>
<tr>
<td>3</td>
<td>9.1</td>
</tr>
<tr>
<td>6</td>
<td>8.5</td>
</tr>
<tr>
<td>9</td>
<td>8.0</td>
</tr>
<tr>
<td>12</td>
<td>7.7</td>
</tr>
<tr>
<td>15</td>
<td>7.5</td>
</tr>
<tr>
<td>18</td>
<td>7.4</td>
</tr>
</tbody>
</table>

41. The region bounded by the curve $y = 1/(1 + e^{-x})$, the $x$- and $y$-axes, and the line $x = 10$ is rotated about the $x$-axis. Use Simpson’s Rule with $n = 10$ to estimate the volume of the resulting solid.

42. The figure shows a pendulum with length $L$ that makes a maximum angle $\theta_0$ with the vertical. Using Newton’s Second Law, it can be shown that the period $T$ (the time for one complete swing) is given by

$$T = \frac{4}{g} \int_{0}^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

where $k = \sin(\frac{1}{2} \theta_0)$ and $g$ is the acceleration due to gravity. If $L = 1$ m and $\theta_0 = 42^\circ$, use Simpson’s Rule with $n = 10$ to find the period.

43. The intensity of light with wavelength $\lambda$ traveling through a diffraction grating with $N$ slits at an angle $\theta_0$ is given by $I(\theta) = N^2 \sin^k \theta_0$, where $k = \frac{\pi N d \sin \theta_0}{\lambda}$ and $d$ is the distance between adjacent slits. A helium-neon laser with wavelength $\lambda = 632.8 \times 10^{-6}$ m is emitting a narrow band of light, given by $-10^{-4} < \theta < 10^{-4}$, through a grating with 10,000 slits spaced $10^{-4}$ m apart. Use the Midpoint Rule with $n = 10$ to estimate the total light intensity $\int_{-10^{-4}}^{10^{-4}} I(\theta) d\theta$ emerging from the grating.

44. Use the Trapezoidal Rule with $n = 10$ to approximate $\int_{0}^{\pi} \cos(\pi x) \, dx$. Compare your result to the actual value. Can you explain the discrepancy?

45. Sketch the graph of a continuous function on $[0, 2]$ for which the Trapezoidal Rule with $n = 2$ is more accurate than the Midpoint Rule.
46. Sketch the graph of a continuous function on \([0, 2]\) for which the right endpoint approximation with \(n = 2\) is more accurate than Simpson’s Rule.

47. If \(f\) is a positive function and \(f''(x) < 0\) for \(a \leq x \leq b\), show that
\[
T_n < \int_a^b f(x) \, dx < M_n
\]

48. Show that if \(f\) is a polynomial of degree 3 or lower, then Simpson’s Rule gives the exact value of \(\int_a^b f(x) \, dx\).

49. Show that \(\frac{1}{2}(T_n + M_n) = T_{2n}\).

50. Show that \(\frac{1}{2}T_n + \frac{3}{4}M_n = S_{2n}\).
Answers

1. (a) \( L_2 = 6, R_2 = 12, M_2 = 9.6 \)  
(b) \( L_2 \) is an underestimate, \( R_2 \) and \( M_2 \) are overestimates. 
(c) \( T_2 < I < R_2 \)  
(d) \( L_2 < I < M_2 < R_2 \) 

2. (a) \( T_4 \approx 0.895759 \) (underestimate) 
(b) \( M_4 \approx 0.908907 \) (overestimate); \( T_4 < I < M_4 \) 

3. (a) \( \int_{E_1}^M \) (overestimate); \( T_4 < I < M_4 \) 
(b) \( S_{10} = 0.805974; E_M = -0.001879 \) 

7. (a) \( 1.506361 \)  
(b) \( 1.518362 \)  
(c) \( 1.511519 \) 

9. (a) \( 2.660833 \)  
(b) \( 2.664377 \)  
(c) \( 2.663244 \) 

11. (a) \( 2.591334 \)  
(b) \( 2.681406 \)  
(c) \( 2.631976 \) 

15. (a) \( -0.495333 \)  
(b) \( -0.543321 \)  
(c) \( -0.526123 \) 

17. (a) \( 8.363853 \)  
(b) \( 8.163298 \)  
(c) \( 8.235114 \) 

19. (a) \( T_4 = 0.902333, M_4 = 0.905620 \) 
(b) \( |E_T| < 0.0078, |E_M| < 0.0039 \) 
(c) \( n = 71 \) for \( T_n, n = 50 \) for \( M_n \) 

21. (a) \( T_{10} = 1.983524, E_T = 0.016476; M_{10} = 2.008248, E_M = -0.008248; S_{10} = 2.000110, E_s = -0.000110 \) 
(b) \( |E_T| < 0.025839, |E_M| < 0.012919, |E_s| < 0.000170 \) 
(c) \( n = 509 \) for \( T_n, n = 360 \) for \( M_n, n = 22 \) for \( S_n \) 

23. (a) \( 2.8 \)  
(b) \( 7.954926518 \)  
(c) \( 0.2894 \) 
(d) \( 7.954926521 \)  
(e) The actual error is much smaller. 
(f) \( 10.9 \)  
(g) \( 7.953789422 \)  
(h) \( 0.0593 \) 
(i) The actual error is smaller.  
(j) \( n \geq 50 \) 

25. | \( n \) | \( L_n \) | \( R_n \) | \( T_n \) | \( M_n \) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.742943</td>
<td>1.286599</td>
<td>1.014771</td>
<td>0.992621</td>
</tr>
<tr>
<td>10</td>
<td>0.867782</td>
<td>1.139610</td>
<td>1.003696</td>
<td>0.998152</td>
</tr>
<tr>
<td>20</td>
<td>0.932967</td>
<td>1.068881</td>
<td>1.000924</td>
<td>0.999538</td>
</tr>
</tbody>
</table>

27. | \( n \) | \( T_n \) | \( M_n \) | \( S_n \) |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>6.695473</td>
<td>6.252572</td>
<td>6.403292</td>
</tr>
<tr>
<td>12</td>
<td>6.474023</td>
<td>6.363008</td>
<td>6.400206</td>
</tr>
</tbody>
</table>

29. (a) \( 19 \)  
(b) \( 18.6 \)  
(c) \( 18.6 \) 

31. (a) \( 14.4 \)  
(b) \( \frac{5}{2} \) 

33. 70.8°F  
35. 37.73 ft/s  
37. 10,177 megawatt-hours 

39. (a) \( 190 \)  
(b) \( 828 \) 

41. 28  
43. 59.4 

45. 

Observations are the same as after Example 1.
Solutions

1. (a) \( \Delta x = (b - a)/n = (4 - 0)/2 = 2 \)

\[
L_2 = \sum_{i=1}^{2} f(x_i) \Delta x = f(x_0) \cdot 2 + f(x_1) \cdot 2 = 2[f(0) + f(2)] = 2(0.5 + 2.5) = 6
\]

\[
R_2 = \sum_{i=1}^{2} f(x_i) \Delta x = f(x_1) \cdot 2 + f(x_2) \cdot 2 = 2[f(2) + f(4)] = 2(2.5 + 3.5) = 12
\]

\[
M_2 = \sum_{i=1}^{2} f(\pi) \Delta x = f(\pi_1) \cdot 2 + f(\pi_2) \cdot 2 = 2[f(1) + f(3)] \approx 2(1.6 + 3.2) = 9.6
\]

\( L_2 \) is an underestimate, since the area under the small rectangles is less than the area under the curve, and \( R_2 \) is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that \( M_2 \) is an overestimate, though it is fairly close to \( I \). See the solution to Exercise 47 for a proof of the fact that if \( f \) is concave down on \([a, b]\), then the Midpoint Rule is an overestimate of \( \int_a^b f(x) \, dx \).

(b) 

\[
T_2 = \left( \frac{1}{2} \Delta x \right)[f(x_0) + 2f(x_1) + f(x_2)] = \frac{1}{2}[f(0) + 2f(2) + f(4)] = 0.5 + 2(2.5) + 3.5 = 9.
\]

This approximation is an underestimate, since the graph is concave down. Thus, \( T_2 = 9 < I \). See the solution to Exercise 47 for a general proof of this conclusion.

(d) For any \( n \), we will have \( L_n < T_n < I < M_n < R_n \).

3. \( f(x) = \cos(x^2) \), \( \Delta x = \frac{b - a}{n} = \frac{1}{4} = \frac{1}{4} \)

(a) \( T_4 = \frac{1}{4} \sum_{i=0}^{3} f \left( \frac{i}{4} \right) = \frac{1}{4}[f(0) + 2f(\frac{1}{4}) + 2f(\frac{2}{4}) + f(1)] \approx 0.895759 \)

(b) \( M_4 = \frac{1}{4} \sum_{i=0}^{3} f \left( \frac{i}{4} \right) + f \left( \frac{5}{4} \right) + f \left( \frac{9}{4} \right) + f \left( \frac{13}{4} \right) \approx 0.908907 \)

The graph shows that \( f \) is concave down on \([0, 1]\). So \( T_4 \) is an underestimate and \( M_4 \) is an overestimate. We can conclude that \( 0.895759 < \int_0^1 \cos(x^2) \, dx < 0.908907 \).

5. \( f(x) = \frac{x}{1 + x^2} \), \( \Delta x = \frac{b - a}{n} \)

\[
M_{10} = \frac{1}{10} \left[ f \left( \frac{1}{10} \right) + f \left( \frac{2}{10} \right) + \cdots + f \left( \frac{10}{10} \right) \right] \approx 0.806598
\]

(b) \( S_{10} = \frac{1}{5} \left[ f(0) + 4f \left( \frac{1}{5} \right) + 2f \left( \frac{2}{5} \right) + 4f \left( \frac{3}{5} \right) + 2f \left( \frac{4}{5} \right) + \cdots + f(2) \right] \approx 0.804779 \)

Actual: \( I = \int_0^1 \frac{x}{1 + x^2} \, dx = \left[ \frac{1}{2} \ln |1 + x^2| \right]^2_0 \ [u = 1 + x^2, \, du = 2x \, dx] \)

\[
= \frac{1}{2} \ln 5 - \frac{1}{2} \ln 1 = \frac{1}{2} \ln 5 \approx 0.804719
\]

Errors: \( E_M = \text{actual} - M_{10} = I - M_{10} \approx -0.001879 \)

\( E_S = \text{actual} - S_{10} = I - S_{10} \approx -0.000060 \)
7. \( f(x) = \sqrt{x^3-1} \), \( \Delta x = \frac{b-a}{n} = \frac{2-1}{10} = \frac{1}{10} \)

(a) \( T_{10} = \frac{1}{10} \left[ f(1) + 2f(1.1) + 2f(1.2) + 2f(1.3) + 2f(1.4) + 2f(1.5) + 2f(1.6) + 2f(1.7) + 2f(1.8) + 2f(1.9) + f(2) \right] \approx 1.506361 \)

(b) \( M_{10} = \frac{1}{10} \left[ f(1.05) + f(1.15) + f(1.25) + f(1.35) + f(1.45) + f(1.55) + f(1.65) + f(1.75) + f(1.85) + f(1.95) \right] \approx 1.518362 \)

(c) \( S_{10} = \frac{1}{10} \left[ f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + 4f(1.5) + 2f(1.6) + 4f(1.7) + 2f(1.8) + 4f(1.9) + f(2) \right] \approx 1.511519 \)

9. \( f(x) = \frac{e^x}{1 + x^2} \), \( \Delta x = \frac{b-a}{n} = \frac{2-0}{10} = \frac{1}{5} \)

(a) \( T_{10} = \frac{1}{5} \left[ f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + 2f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2) \right] \approx 2.660833 \)

(b) \( M_{10} = \frac{1}{5} \left[ f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9) + f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9) \right] \approx 2.664377 \)

(c) \( S_{10} = \frac{1}{5} \left[ f(0) + 4f(0.2) + 2f(0.4) + 4f(0.6) + 2f(0.8) + 4f(1) + 2f(1.2) + 4f(1.4) + 2f(1.6) + 4f(1.8) + f(2) \right] \approx 2.663244 \)

11. \( f(x) = \sqrt{\ln x} \), \( \Delta x = \frac{\sqrt{1-x}}{2} = \frac{1}{2} \)

(a) \( T_6 = \frac{1}{2} \left[ f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + f(4) \right] \approx 2.501334 \)

(b) \( M_6 = \frac{1}{2} \left[ f(1.25) + f(1.75) + f(2.25) + f(2.75) + f(3.25) + f(3.75) \right] \approx 2.681046 \)

(c) \( S_6 = \frac{1}{2} \left[ f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4) \right] \approx 2.631976 \)

13. \( f(t) = e^{\sqrt{t}} \sin t \), \( \Delta t = \frac{b-a}{n} = \frac{1}{3} \)

(a) \( T_8 = \frac{1}{3} \left[ f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3) + 2f(3.5) + f(4) \right] \approx 4.513618 \)

(b) \( M_8 = \frac{1}{3} \left[ f(0.25) + f(0.75) + f(1.25) + f(1.75) + f(2.25) + f(2.75) + f(3.25) + f(3.75) \right] \approx 4.748256 \)

(c) \( S_8 = \frac{1}{3} \left[ f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4) \right] \approx 4.675111 \)

15. \( f(x) = \frac{\cos x}{x} \), \( \Delta x = \frac{b-a}{n} = \frac{1}{2} \)

(a) \( T_8 = \frac{1}{2} \left[ f(1) + 2f(1.5) + 2f(2) + \cdots + 2f(4) + 2f(4.5) + f(5) \right] \approx -0.495333 \)

(b) \( M_8 = \frac{1}{2} \left[ f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3) + f(3.5) + f(4) \right] \approx -0.543321 \)

(c) \( S_8 = \frac{1}{2} \left[ f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + 2f(4) + 4f(4.5) + f(5) \right] \approx -0.526123 \)
17. \( f(x) = e^{-x}, \Delta x = \frac{1 - (-1)}{10} = \frac{1}{5} \)

(a) \( T_{10} = \frac{1}{5} \left[ f(-1) + 2f(-0.8) + 2f(-0.6) + 2f(-0.4) + 2f(-0.2) + f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1) \right] \)
\[ \approx 8.363853 \]

(b) \( M_{10} = \frac{1}{5} \left[ f(-0.9) + f(-0.7) + f(-0.5) + f(-0.3) + f(-0.1) + f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9) \right] \)
\[ \approx 8.163298 \]

(c) \( S_{10} = \frac{1}{5} \left[ f(-1) + 4f(-0.8) + 2f(-0.6) + 4f(-0.4) + 2f(-0.2) + 4f(0) + 2f(0.2) + 4f(0.4) + 2f(0.6) + 4f(0.8) + f(1) \right] \)
\[ \approx 8.235114 \]

19. \( f(x) = \cos(x^2), \Delta x = \frac{\sqrt{2} - 0}{8} = \frac{1}{8} \)

(a) \( T_8 = \frac{1}{\sqrt{2}} \left\{ f(0) + 2f\left(\frac{\sqrt{2}}{8}\right) + \cdots + f\left(\frac{7\sqrt{2}}{8}\right) \right\} + f(1) \approx 0.902333 \)

\( M_8 = \frac{1}{\sqrt{2}} \left[ f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f\left(\frac{15}{8}\right) \right] = 0.905620 \)

(b) \( f(x) = \cos(x^2), f'(x) = -2x \sin(x^2), f''(x) = -2 \sin(x^2) - 4x^2 \cos(x^2) \).

For \( 0 \leq x \leq 1 \), sin and cos are positive, so \( |f''(x)| = 2 \sin(x^2) + 4x^2 \cos(x^2) \leq 2 \cdot 1 + 4 \cdot 1 \cdot 1 = 6 \) since \( \sin(x^2) \leq 1 \) and \( \cos(x^2) \leq 1 \) for all \( x \), and \( x^2 \leq 1 \) for \( 0 \leq x \leq 1 \). So for \( n = 8 \), we take \( K = 6, a = 0, b = 1 \) in Theorem 3, to get
\[ |E_T| \leq 6 \cdot 1^3/(12 \cdot 2^3) = \frac{\pi}{192} = 0.0078125 \text{ and } |E_M| \leq \frac{\pi}{256} = 0.00390625. \text{ [A better estimate is obtained by noting from a graph of } f'' \text{ that } |f''(x)| \leq 4 \text{ for } 0 \leq x \leq 1.] \]

(c) Take \( K = 6 \) [as in part (b)] in Theorem 3.
\[ |E_T| \leq \frac{K(b-a)^3}{12n^2} \leq 0.0001 \iff \frac{6(1-0)^3}{12n^2} \leq 10^{-4} \iff \]
\[ \frac{1}{2n^2} \leq \frac{1}{10^4} \iff 2n^2 \geq 10^4 \iff n^2 \geq 5000 \iff n \geq 71. \text{ Take } n = 71 \text{ for } T_n. \text{ For } E_M, \text{ again take } K = 6 \text{ in Theorem 3 to get } |E_M| \leq 10^{-4} \iff 4n^2 \geq 10^4 \iff n^2 \geq 2500 \iff n \geq 50. \text{ Take } n = 50 \text{ for } M_n. \]

21. \( f(x) = \sin x, \Delta x = \frac{\pi - 0}{10} = \frac{\pi}{10} \)

(a) \( T_{10} = \frac{\pi}{10} \left\{ f(0) + 2f\left(\frac{\pi}{10}\right) + \cdots + 2f\left(\frac{9\pi}{10}\right) \right\} + f(\pi) \approx 1.983524 \)

\( M_{10} = \frac{\pi}{10} \left[ f\left(\frac{\pi}{10}\right) + f\left(\frac{3\pi}{10}\right) + \cdots + f\left(\frac{9\pi}{10}\right) \right] \approx 2.008248 \)

\( S_{10} = \frac{\pi}{10} \left\{ f(0) + 4f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + 4f\left(\frac{3\pi}{10}\right) + \cdots + 4f\left(\frac{9\pi}{10}\right) \right\} + f(\pi) \approx 2.000110 \)

Since \( I = \int_0^\pi \sin x \, dx = [-\cos x]_0^\pi = 1 - (-1) = 2, \) \( E_T = I - T_{10} \approx 0.016476, \) \( E_M = I - M_{10} \approx -0.008248, \) and \( E_S = I - S_{10} \approx -0.000110. \)

(b) \( f(x) = \sin x \Rightarrow |f^{(n)}(x)| \leq 1, \) so take \( K = 1 \) for all error estimates.
\[ |E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{(1-0)^3}{12(10)^2} = \frac{\pi^3}{1200} \approx 0.025839. \]
\[ |E_M| \leq \frac{|E_T|}{2} = \frac{\pi^3}{2400} \approx 0.012919. \]
\[ |E_S| \leq \frac{K(b-a)^3}{180n^4} = \frac{(1-0)^3}{180(10)^4} = \frac{\pi^5}{1,500,000} \approx 0.000170. \]

The actual error is about 64% of the error estimate in all three cases.
(c) \( |E_T| \leq 0.00001 \quad \Leftrightarrow \quad \frac{\pi^3}{12n^2} \leq \frac{1}{10^6} \quad \Leftrightarrow \quad n^2 \geq \frac{10^6\pi^3}{12} \quad \Rightarrow \quad n \geq 508.3. \) Take \( n = 509 \) for \( T_n. \)

\( |E_M| \leq 0.00001 \quad \Leftrightarrow \quad \frac{\pi^3}{24n^2} \leq \frac{1}{10^6} \quad \Leftrightarrow \quad n^2 \geq \frac{10^6\pi^3}{24} \quad \Rightarrow \quad n \geq 359.4. \) Take \( n = 360 \) for \( M_n. \)

\( |E_S| \leq 0.00001 \quad \Leftrightarrow \quad \frac{\pi^5}{180n^4} \leq \frac{1}{10^6} \quad \Leftrightarrow \quad n^4 \geq \frac{10^6\pi^5}{180} \quad \Rightarrow \quad n \geq 20.3. \)

Take \( n = 22 \) for \( S_n \) (since \( n \) must be even).

23. (a) Using a CAS, we differentiate \( f(x) = e^{\cos x} \) twice, and find that

\( f''(x) = e^{\cos x}(\sin^2 x - \cos x). \) From the graph, we see that the maximum value of \( |f''(x)| \) occurs at the endpoints of the interval \([0, 2\pi].\)

Since \( f''(0) = -e, \) we can use \( K = e \) or \( K = 2.8. \)

(b) A CAS gives \( M_{10} \approx 7.954926518. \) (In Maple, use \texttt{Student[Calculus1][RiemannSum]} or \texttt{Student[Calculus1][ApproximateInt]}.)

(c) Using Theorem 3 for the Midpoint Rule, with \( K = e, \) we get \( |E_M| \leq \frac{e(2\pi - 0)^3}{24 \cdot 10^6} \approx 0.280945995. \)

With \( K = 2.8, \) we get \( |E_M| \leq \frac{2.8(2\pi - 0)^3}{24 \cdot 10^6} = 0.289391916. \)

(d) A CAS gives \( I \approx 7.954926521. \)

(e) The actual error is only about \( 3 \times 10^{-9}, \) much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph

\( f^{(4)}(x) = e^{\cos x}(\sin^4 x - 6 \sin^2 x \cos x + 3 - 7 \sin^2 x + \cos x). \)

From the graph, we see that the maximum value of \( |f^{(4)}(x)| \) occurs at the endpoints of the interval \([0, 2\pi].\) Since \( f^{(4)}(0) = 4e, \) we can use \( K = 4e \) or \( K = 10.9. \)

(g) A CAS gives \( S_{10} \approx 7.953789422. \) (In Maple, use \texttt{Student[Calculus1][ApproximateInt]}.)

(h) Using Theorem 4 with \( K = 4e, \) we get \( |E_S| \leq \frac{4e(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059153618. \)

With \( K = 10.9, \) we get \( |E_S| \leq \frac{10.9(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059929814. \)

(i) The actual error is about \( 7.954926521 - 7.953789422 \approx 0.00114. \) This is quite a bit smaller than the estimate in part (h), though the difference is not nearly as great as it was in the case of the Midpoint Rule.

(j) To ensure that \( |E_S| \leq 0.0001, \) we use Theorem 4: \( |E_S| \leq \frac{4e(2\pi - 0)^5}{180 \cdot n^4} \leq 0.0001 \quad \Rightarrow \quad \frac{4e(2\pi)^5}{180 \cdot 0.0001} \leq n^4 \quad \Rightarrow \quad n^4 \geq 5,915,362 \quad \Leftrightarrow \quad n \geq 49.3. \) So we must take \( n \geq 50 \) to ensure that \( |I - S_n| \leq 0.0001. \)

\((K = 10.9 \text{ leads to the same value of } n.)\)
25. $I = \int_0^1 x e^x \, dx = [(x - 1)e^x]_0^1$ [by parts] $= 0 - (-1) = 1$, $f(x) = xe^x$, $\Delta x = 1/n$

For $n = 5$:
- $L_5 = \frac{1}{5}[f(0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \approx 0.742943$
- $R_5 = \frac{1}{5}[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \approx 1.286599$
- $T_5 = \frac{1}{5}[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)] \approx 1.014771$
- $M_5 = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \approx 0.992621$

$E_L = I - L_5 \approx 1 - 0.742943 = 0.257057$
$E_R \approx 1 - 1.286599 = -0.286599$
$E_T \approx 1 - 1.014771 = -0.014771$
$E_M \approx 1 - 0.992621 = 0.007379$

For $n = 10$:
- $L_{10} = \frac{1}{10}[f(0) + f(0.1) + f(0.2) + \cdots + f(0.9)] \approx 0.867782$
- $R_{10} = \frac{1}{10}[f(0.1) + f(0.2) + \cdots + f(0.9) + f(1)] \approx 1.139610$
- $T_{10} = \frac{1}{10}[f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)] \approx 1.003696$
- $M_{10} = \frac{1}{10}[f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95)] \approx 0.998152$

$E_L = I - L_{10} \approx 1 - 0.867782 = 0.132218$
$E_R \approx 1 - 1.139610 = -0.139610$
$E_T \approx 1 - 1.003696 = -0.003696$
$E_M \approx 1 - 0.998152 = 0.001848$

For $n = 20$:
- $L_{20} = \frac{1}{20}[f(0) + f(0.05) + f(0.10) + \cdots + f(0.95)] \approx 0.932967$
- $R_{20} = \frac{1}{20}[f(0.05) + f(0.10) + \cdots + f(0.95) + f(1)] \approx 1.068881$
- $T_{20} = \frac{1}{20}[f(0) + 2[f(0.05) + f(0.10) + \cdots + f(0.95)] + f(1)] \approx 1.000924$
- $M_{20} = \frac{1}{20}[f(0.025) + f(0.075) + f(0.125) + \cdots + f(0.975)] \approx 0.999538$

$E_L = I - L_{20} \approx 1 - 0.932967 = 0.067033$
$E_R \approx 1 - 1.068881 = -0.068881$
$E_T \approx 1 - 1.000924 = -0.000924$
$E_M \approx 1 - 0.999538 = 0.000462$

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Observations:
1. $E_L$ and $E_R$ are always opposite in sign, as are $E_T$ and $E_M$.
2. As $n$ is doubled, $E_L$ and $E_R$ are decreased by about a factor of 2, and $E_T$ and $E_M$ are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of $n$ increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.
27. \( I = \int_0^1 x^4 \, dx = \left[ \frac{1}{5} x^5 \right]_0^1 = \frac{32}{5} - 0 = 6.4, f(x) = x^4, \Delta x = \frac{2 - 0}{n} = \frac{2}{n} \)

\( n = 6: \)
\[ T_6 = \frac{2}{3} \left[ f(0) + 2 f\left( \frac{1}{3} \right) + f\left( \frac{2}{3} \right) + f\left( \frac{4}{3} \right) + f\left( \frac{5}{3} \right) + f(2) \right] \approx 6.695473 \]
\[ M_6 = \frac{2}{3} \left[ f\left( \frac{1}{3} \right) + f\left( \frac{2}{3} \right) + f\left( \frac{4}{3} \right) + f\left( \frac{5}{3} \right) + f(1) + f(2) \right] \approx 6.252572 \]
\[ S_6 = \frac{2}{3} \left[ f(0) + 4 f\left( \frac{1}{3} \right) + 2 f\left( \frac{2}{3} \right) + 4 f\left( \frac{4}{3} \right) + 2 f\left( \frac{5}{3} \right) + f(2) \right] \approx 6.403292 \]
\[ E_T = I - T_6 \approx 6.4 - 6.695473 = -0.295473 \]
\[ E_M \approx 6.4 - 6.252572 = 0.147428 \]
\[ E_S \approx 6.4 - 6.403292 = -0.003292 \]

\( n = 12: \)
\[ T_{12} = \frac{2}{3} \left[ f(0) + 2 f\left( \frac{1}{3} \right) + f\left( \frac{2}{3} \right) + f\left( \frac{4}{3} \right) + \cdots + f\left( \frac{11}{3} \right) + f(2) \right] \approx 6.74023 \]
\[ M_{12} = \frac{2}{3} \left[ f\left( \frac{1}{3} \right) + f\left( \frac{2}{3} \right) + f\left( \frac{4}{3} \right) + \cdots + f\left( \frac{11}{3} \right) + f(1) + f(2) \right] \approx 6.363008 \]
\[ S_{12} = \frac{2}{3} \left[ f(0) + 4 f\left( \frac{1}{3} \right) + 2 f\left( \frac{2}{3} \right) + 4 f\left( \frac{4}{3} \right) + 2 f\left( \frac{5}{3} \right) + \cdots + 4 f\left( \frac{11}{3} \right) + f(2) \right] \approx 6.402006 \]
\[ E_T = I - T_{12} \approx 6.4 - 6.74023 = -0.074023 \]
\[ E_M \approx 6.4 - 6.363008 = 0.036992 \]
\[ E_S \approx 6.4 - 6.402006 = -0.002006 \]

**Observations:**
1. \( E_T \) and \( E_M \) are opposite in sign and decrease by a factor of about 4 as \( n \) is doubled.
2. The Simpson’s approximation is much more accurate than the Midpoint and Trapezoidal approximations, and \( E_S \) seems to decrease by a factor of about 16 as \( n \) is doubled.

29. \( \Delta x = (b - a)/n = (6 - 0)/6 = 1 \)

(a) \( T_6 = \frac{1}{3} \left[ f(0) + 2 f(1) + 2 f(2) + 2 f(3) + 2 f(4) + 2 f(5) + f(6) \right] \)
\[ \approx \frac{1}{3} [3 + 2(5) + 2(4) + 2(2) + 2(2.8) + 2(4) + 1] \]
\[ = \frac{1}{3}(39.6) = 19.8 \]
(b) \( M_6 = \Delta x [f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)] \)
\[ \approx 1 [4.5 + 4.7 + 2.6 + 2.2 + 3.4 + 3.2] \]
\[ = 20.6 \]
(c) \( S_6 = \frac{1}{3} \left[ f(0) + 4 f(1) + 2 f(2) + 4 f(3) + 2 f(4) + 4 f(5) + f(6) \right] \)
\[ \approx \frac{1}{3} [3 + 4(5) + 2(4) + 2(2) + 2(2.8) + 4(4) + 1] \]
\[ = \frac{1}{3}(61.6) = 20.53 \]

31. \( \int_1^3 f(x) \, dx \approx M_4 = \frac{5 - 1}{4} [f(1.5) + f(2.5) + f(3.5) + f(4.5)] = 1(2.9 + 3.6 + 4.0 + 3.9) = 14.4 \)
(b) \(-2 \leq f''(x) \leq 3 \Rightarrow |f''(x)| \leq 3 \Rightarrow K = 3\), since \(|f''(x)| \leq K\). The error estimate for the Midpoint Rule is

\[
|E_M| \leq \frac{K(b-a)^3}{24n^2} = \frac{3(5 - 1)^3}{24(4)^2} = \frac{1}{2}.
\]

33. \(T_{ave} = \frac{1}{24-0} \int_0^{24} T(t) \, dt \approx \frac{1}{24} s_{12} = \frac{1}{24} \frac{24 - 0}{3(12)} [T(0) + 4T(2) + 2T(4) + 4T(6) + 2T(8) + 4T(10) + 2T(12) + 4T(14) + 2T(16) + 4T(18) + 2T(20) + 4T(22) + T(24)]

\[\approx \frac{1}{24} [67 + 4(65) + 2(62) + 4(58) + 2(56) + 4(61) + 2(63) + 4(68) + 2(71) + 4(69) + 2(67) + 4(66) + 64] = \frac{1}{24} (2317) = 64.36^\circ F.\]

The average temperature was about 64.4\(^\circ\)F.

35. By the Net Change Theorem, the increase in velocity is equal to \(\int_0^6 a(t) \, dt\). We use Simpson’s Rule with \(n = 6\) and \(\Delta t = (6 - 0)/6 = 1\) to estimate this integral:

\[
\int_0^6 a(t) \, dt \approx S_6 = \frac{1}{3} \left[ a(0) + 4a(1) + 2a(2) + 4a(3) + 2a(4) + 4a(5) + a(6) \right]
\]

\[\approx \frac{1}{3} [0 + 4(0.5) + 2(4.1) + 4(9.8) + 2(12.9) + 4(9.5) + 0] = \frac{1}{3} (113.2) = 37.73 \text{ ft/s}.
\]

37. By the Net Change Theorem, the energy used is equal to \(\int_0^6 P(t) \, dt\). We use Simpson’s Rule with \(n = 12\) and \(\Delta t = \frac{6-0}{12} = \frac{1}{2}\) to estimate this integral:

\[
\int_0^6 P(t) \, dt \approx S_{12} = \frac{1}{3} \left[ P(0) + 4P(0.5) + 2P(1) + 4P(1.5) + 2P(2) + 4P(2.5) + 2P(3) + 4P(3.5) + 2P(4) + 4P(4.5) + 2P(5) + 4P(5.5) + P(6) \right]
\]

\[= \frac{1}{3} \left[ 1814 + 4(1735) + 2(1686) + 4(1646) + 2(1637) + 4(1609) + 2(1604) + 4(1611) + 2(1621) + 4(1666) + 2(1745) + 4(1886) + 2052 \right]
\]

\[= \frac{1}{3} (61,064) = 10,177.3 \text{ megawatt-hours}.
\]

39. (a) Let \(y = f(x)\) denote the curve. Using disks, \(V = \int_2^{10} \pi [f(x)]^2 \, dx = \pi \int_2^{10} g(x) \, dx = \pi I_1\).

Now use Simpson’s Rule to approximate \(I_1\):

\[
I_1 \approx S_6 = \frac{10 - 2}{6} \left[ g(2) + 4g(3) + 2g(4) + 4g(5) + 2g(6) + 4g(7) + g(8) \right]
\]

\[\approx \frac{1}{3} [0^2 + 4(1.5)^2 + 2(1.9)^2 + 4(2.2)^2 + 2(3.0)^2 + 4(3.8)^2 + 2(4.0)^2 + 4(3.1)^2 + 0^2] = \frac{1}{3} (181.78)
\]

Thus, \(V \approx \pi \cdot \frac{1}{3} (181.78) \approx 190.4\) or 190 cubic units.

(b) Using cylindrical shells, \(V = \int_2^{10} 2\pi x f(x) \, dx = 2\pi \int_2^{10} x f(x) \, dx = 2\pi I_1\).

Now use Simpson’s Rule to approximate \(I_1\):

\[
I_1 \approx S_6 = \frac{10 - 2}{6} \left[ f(2) + 4 \cdot 3f(3) + 2 \cdot 4f(4) + 4 \cdot 5f(5) + 2 \cdot 6f(6) + 4 \cdot 7f(7) + 2 \cdot 8f(8) + 4 \cdot 9f(9) + 10f(10) \right]
\]

\[\approx \frac{1}{3} [2(0) + 12(1.5) + 8(1.9) + 20(2.2) + 12(3.0) + 28(3.8) + 16(4.0) + 36(3.1) + 10(0)] = \frac{1}{3} (395.2)
\]

Thus, \(V \approx \pi \cdot \frac{1}{3} (181.78) \approx 190.4\) or 190 cubic units.
41. Using disks, \( V = \int_1^5 \pi (e^{-1/x})^2 \, dx = \pi \int_1^5 e^{-2/x} \, dx = \pi I_1 \). Now use Simpson’s Rule with \( f(x) = e^{-2/x} \) to approximate

\[ I_1 \approx S_8 = \frac{\Delta x}{3} \sum_{i=1}^{8} f(c_i) \]

where \( c_i = 1 + \frac{i-1}{7} \). Thus, \( V \approx \pi \cdot \frac{3}{14}(11.4566) \approx 6.0 \) cubic units.

43. \( I(\theta) = \frac{N^2 \sin^2 k}{k^2} \), where \( k = \frac{\pi N d \sin \theta}{\lambda} \), \( N = 10,000, d = 10^{-4}, \) and \( \lambda = 632.8 \times 10^{-9} \). So \( I(\theta) = \frac{(10^4)^2 \sin^2 k}{k^2} \), where \( k = \frac{\pi(10^4)(10^{-4}) \sin \theta}{632.8 \times 10^{-9}} \). Now \( n = 10 \) and \( \Delta \theta = \frac{10^{-6} - (-10^{-6})}{10} = 2 \times 10^{-7} \), so \( M_{10} = 2 \times 10^{-7}[I(-0.0000009) + I(-0.0000007) + \cdots + I(0.0000009)] \approx 59.4 \).

45. Consider the function \( f \) whose graph is shown. The area \( \int_0^3 f(x) \, dx \)

is close to 2. The Trapezoidal Rule gives

\[ T_2 = \frac{2 - 0}{2}[f(0) + 2f(1) + f(2)] = \frac{1}{2}[1 + 2 \cdot 1 + 1] = 2. \]

The Midpoint Rule gives \( M_2 = \frac{2 - 0}{2}[f(0.5) + f(1.5)] = 1[0 + 0] = 0 \), so the Trapezoidal Rule is more accurate.

47. Since the Trapezoidal and Midpoint approximations on the interval \([a, b]\) are the sums of the Trapezoidal and Midpoint approximations on the subintervals \([x_{i-1}, x_i], i = 1, 2, \ldots, n\), we can focus our attention on one such interval. The condition \( f''(x) < 0 \) for \( a \leq x \leq b \) means that the graph of \( f \) is concave down as in Figure 5. In that figure, \( T_n \) is the area of the trapezoid \( AQRD \), \( \int_a^b f(x) \, dx \) is the area of the region \( AQPRD \), and \( M_n \) is the area of the trapezoid \( ABCD \), so \( T_n < \int_a^b f(x) \, dx < M_n \). In general, the condition \( f'' < 0 \) implies that the graph of \( f \) on \([a, b]\) lies above the chord joining the points \((a, f(a))\) and \((b, f(b))\). Thus, \( \int_a^b f(x) \, dx > T_n \). Since \( M_n \) is the area under a tangent to the graph, and since \( f'' < 0 \) implies that the tangent lies above the graph, we also have \( M_n > \int_a^b f(x) \, dx \). Thus, \( T_n < \int_a^b f(x) \, dx < M_n \).

49. \( T_n = \frac{1}{2} \Delta x \sum_{i=1}^{n-1} f(x_i) \) and

\[ M_n = \Delta x \left[ f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n) \right], \]

where \( x_i = \frac{1}{2}(x_{i-1} + x_i) \). Now

\[ T_{2n} = \frac{1}{4} \left( \frac{3}{2} \Delta x \right) \sum_{i=1}^{n-1} f(x_i), \]

so

\[ \frac{1}{2}(T_n + M_n) = \frac{1}{2}T_n + \frac{1}{2}M_n \]

\[ = \frac{1}{2} \Delta x \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2} \Delta x \left[ 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + 2f(x_n) \right] \]

\[ = T_{2n} \]