First-Order Linear Differential Equations

A first-order **linear** differential equation is one that can be put into the form

1 $$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions on a given interval. This type of equation occurs frequently in various sciences, as we will see.

An example of a linear equation is $xy' + y = 2x$ because, for $x \neq 0$, it can be written in the form

2 $$y' + \frac{1}{x}y = 2$$

Notice that this differential equation is not separable because it's impossible to factor the expression for y' as a function of x times a function of y. But we can still solve the equation by noticing, by the Product Rule, that

$$xy' + y = (xy)'$$

and so we can rewrite the equation as

$$(xy)' = 2x$$

If we now integrate both sides of this equation, we get

$$xy = x^2 + C \qquad \text{or} \qquad y = x + \frac{C}{x}$$

If we had been given the differential equation in the form of Equation 2, we would have had to take the preliminary step of multiplying each side of the equation by x.

It turns out that every first-order linear differential equation can be solved in a similar fashion by multiplying both sides of Equation 1 by a suitable function $I(x)$ called an *integrating factor*. We try to find I so that the left side of Equation 1, when multiplied by $I(x)$, becomes the derivative of the product $I(x)y$:

3 $$I(x)\big(y' + P(x)y\big) = \big(I(x)y\big)'$$

If we can find such a function I, then Equation 1 becomes

$$\big(I(x)y\big)' = I(x)\,Q(x)$$

Integrating both sides, we would have

$$I(x)y = \int I(x)\,Q(x)\,dx + C$$

so the solution would be

4 $$y(x) = \frac{1}{I(x)}\left[\int I(x)\,Q(x)\,dx + C\right]$$

To find such an I, we expand Equation 3 and cancel terms:

$$I(x)y' + I(x)\,P(x)y = \big(I(x)y\big)' = I'(x)y + I(x)y'$$

$$I(x)\,P(x) = I'(x)$$

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This is a separable differential equation for I, which we solve as follows:

$$\int \frac{dI}{I} = \int P(x)\,dx$$

$$\ln|I| = \int P(x)\,dx$$

$$I = Ae^{\int P(x)\,dx}$$

where $A = \pm e^C$. We are looking for a particular integrating factor, not the most general one, so we take $A = 1$ and use

5 $$I(x) = e^{\int P(x)\,dx}$$

Thus a formula for the general solution to Equation 1 is provided by Equation 4, where I is given by Equation 5. Instead of memorizing this formula, however, we just remember the form of the integrating factor.

> To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the **integrating factor** $I(x) = e^{\int P(x)\,dx}$ and integrate both sides.

EXAMPLE 1 Solve the differential equation $\dfrac{dy}{dx} + 3x^2y = 6x^2$.

SOLUTION The given equation is linear since it has the form of Equation 1 with $P(x) = 3x^2$ and $Q(x) = 6x^2$. An integrating factor is

$$I(x) = e^{\int 3x^2\,dx} = e^{x^3}$$

Figure 1 shows the graphs of several members of the family of solutions in Example 1. Notice that they all approach 2 as $x \to \infty$.

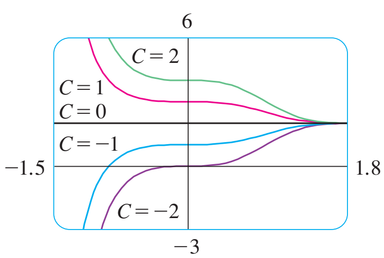

FIGURE 1

Multiplying both sides of the differential equation by e^{x^3}, we get

$$e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = 6x^2e^{x^3}$$

or

$$\frac{d}{dx}(e^{x^3}y) = 6x^2e^{x^3}$$

Integrating both sides, we have

$$e^{x^3}y = \int 6x^2e^{x^3}\,dx = 2e^{x^3} + C$$

$$y = 2 + Ce^{-x^3}$$ ■

EXAMPLE 2 Find the solution of the initial-value problem

$$x^2y' + xy = 1 \qquad x > 0 \qquad y(1) = 2$$

SOLUTION We must first divide both sides by the coefficient of y' to put the differential equation into standard form:

6 $$y' + \frac{1}{x}y = \frac{1}{x^2} \qquad x > 0$$

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The integrating factor is

$$I(x) = e^{\int (1/x)\,dx} = e^{\ln x} = x$$

Multiplication of Equation 6 by x gives

$$xy' + y = \frac{1}{x} \qquad \text{or} \qquad (xy)' = \frac{1}{x}$$

Then

$$xy = \int \frac{1}{x}\,dx = \ln x + C$$

and so

$$y = \frac{\ln x + C}{x}$$

Since $y(1) = 2$, we have

$$2 = \frac{\ln 1 + C}{1} = C$$

Therefore the solution to the initial-value problem is

$$y = \frac{\ln x + 2}{x}$$ ■

The solution of the initial-value problem in Example 2 is shown in Figure 2.

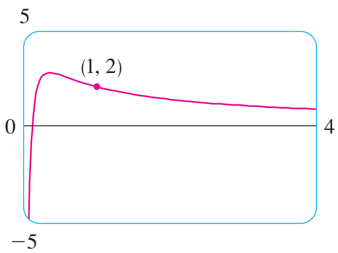

FIGURE 2

EXAMPLE 3 Solve $y' + 2xy = 1$.

SOLUTION The given equation is in the standard form for a linear equation. Multiplying by the integrating factor

$$e^{\int 2x\,dx} = e^{x^2}$$

we get

$$e^{x^2}y' + 2xe^{x^2}y = e^{x^2}$$

or

$$\left(e^{x^2}y\right)' = e^{x^2}$$

Therefore

$$e^{x^2}y = \int e^{x^2}\,dx + C$$

Recall from Section 5.7 that $\int e^{x^2}dx$ can't be expressed in terms of elementary functions. Nonetheless, it's a perfectly good function and we can leave the answer as

$$y = e^{-x^2}\int e^{x^2}\,dx + Ce^{-x^2}$$

Another way of writing the solution is

$$y = e^{-x^2}\int_0^x e^{t^2}\,dt + Ce^{-x^2}$$

(Any number can be chosen for the lower limit of integration.) ■

Even though the solutions of the differential equation in Example 3 are expressed in terms of an integral, they can still be graphed by a computer algebra system (Figure 3).

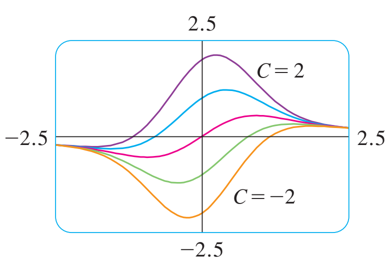

FIGURE 3

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Exercises

1–4 ■ Determine whether the differential equation is linear.

1. $x - y' = xy$

2. $y' + xy^2 = \sqrt{x}$

3. $y' = \dfrac{1}{x} + \dfrac{1}{y}$

4. $y \sin x = x^2 y' - x$

5–14 ■ Solve the differential equation.

5. $y' + y = 1$

6. $y' - y = e^x$

7. $y' = x - y$

8. $4x^3 y + x^4 y' = \sin^3 x$

9. $xy' + y = \sqrt{x}$

10. $y' + y = \sin(e^x)$

11. $\sin x \dfrac{dy}{dx} + (\cos x)y = \sin(x^2)$

12. $x \dfrac{dy}{dx} - 4y = x^4 e^x$

13. $(1 + t)\dfrac{du}{dt} + u = 1 + t, \quad t > 0$

14. $t \ln t \dfrac{dr}{dt} + r = te^t$

15–20 ■ Solve the initial-value problem.

15. $x^2 y' + 2xy = \ln x, \quad y(1) = 2$

16. $t^3 \dfrac{dy}{dt} + 3t^2 y = \cos t, \quad y(\pi) = 0$

17. $t \dfrac{du}{dt} = t^2 + 3u, \quad t > 0, \quad u(2) = 4$

18. $2xy' + y = 6x, \quad x > 0, \quad y(4) = 20$

19. $xy' = y + x^2 \sin x, \quad y(\pi) = 0$

20. $(x^2 + 1)\dfrac{dy}{dx} + 3x(y - 1) = 0, \quad y(0) = 2$

21–22 ■ Solve the differential equation and use a calculator to graph several members of the family of solutions. How does the solution curve change as C varies?

21. $xy' + 2y = e^x$

22. $xy' = x^2 + 2y$

23. A **Bernoulli differential equation** (named after James Bernoulli) is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, show that the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x)$$

24–25 ■ Use the method of Exercise 23 to solve the differential equation.

24. $xy' + y = -xy^2$

25. $y' + \dfrac{2}{x}y = \dfrac{y^3}{x^2}$

26. Solve the second-order equation $xy'' + 2y' = 12x^2$ by making the substitution $u = y'$.

27. Let $P(t)$ be the performance level of someone learning a skill as a function of the training time t. The graph of P is called a *learning curve*. We propose the differential equation

$$\frac{dP}{dt} = k[M - P(t)]$$

as a reasonable model for learning, where k is a positive con-stant. Solve it as a linear differential equation and use your solution to graph the learning curve.

28. Two new workers were hired for an assembly line. Jim processed 25 units during the first hour and 45 units during the second hour. Mark processed 35 units during the first hour and 50 units the second hour. Using the model of Exercise 31 and assuming that $P(0) = 0$, estimate the maximum number of units per hour that each worker is capable of processing.

29. In Section 7.4 we looked at mixing problems in which the volume of fluid remained constant and saw that such problems give rise to separable differentiable equations. (See Exercises 45–48 in that section.) If the rates of flow into and out of the system are different, then the volume is not constant and the resulting differential equation is linear but not separable. Exercise 7.4.44 is an example.

A tank contains 100 L of water. A solution with a salt concentration of 0.4 kg/L is added at a rate of 5 L/min. The solution is kept mixed and is drained from the tank at a rate of 3 L/min. If $y(t)$ is the amount of salt (in kilograms) after t minutes, show that y satisfies the differential equation

$$\frac{dy}{dt} = 2 - \frac{3y}{100 + 2t}$$

Solve this equation and find the concentration after 20 minutes.

30. A tank with a capacity of 400 L is full of a mixture of water and chlorine with a concentration of 0.05 g of chlorine per liter. In order to reduce the concentration of chlorine, fresh water is pumped into the tank at a rate of 4 L/s. The mixture is kept stirred and is pumped out at a rate of 10 L/s. Find the amount of chlorine in the tank as a function of time.

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31. An object with mass m is dropped from rest and we assume that the air resistance is proportional to the speed of the object. If $s(t)$ is the distance dropped after t seconds, then the speed is $v = s'(t)$ and the acceleration is $a = v'(t)$. If g is the acceleration due to gravity, then the downward force on the object is $mg - cv$, where c is a positive constant, and Newton's Second Law gives

$$m\frac{dv}{dt} = mg - cv$$

(a) Solve this as a linear equation to show that

$$v = \frac{mg}{c}(1 - e^{-ct/m})$$

(b) What is the limiting velocity?
(c) Find the distance the object has fallen after t seconds.

32. If we ignore air resistance, we can conclude that heavier objects fall no faster than lighter objects. But if we take air resistance into account, our conclusion changes. Use the expression for the velocity of a falling object in Exercise 31(a) to find dv/dm and show that heavier objects *do* fall faster than lighter ones.

33. (a) Show that the substitution $z = 1/P$ transforms the logistic differential equation $P' = kP(1 - P/M)$ into the linear differential equation

$$z' + kz = \frac{k}{M}$$

(b) Solve the linear differential equation in part (a) and thus obtain an expression for $P(t)$. Compare with Equation 9.4.7.

34. To account for seasonal variation in the logistic differential equation, we could allow k and M to be functions of t:

$$\frac{dP}{dt} = k(t)P\left(1 - \frac{P}{M(t)}\right)$$

(a) Verify that the substitution $z = 1/P$ transforms this equation into the linear equation

$$\frac{dz}{dt} + k(t)z = \frac{k(t)}{M(t)}$$

(b) Write an expression for the solution of the linear equation in part (a) and use it to show that if the carrying capacity M is constant, then

$$P(t) = \frac{M}{1 + CMe^{-\int k(t)\,dt}}$$

Deduce that if $\int_0^\infty k(t)\,dt = \infty$, then $\lim_{t\to\infty} P(t) = M$. [This will be true if $k(t) = k_0 + a\cos bt$ with $k_0 > 0$, which describes a positive intrinsic growth rate with a periodic seasonal variation.]

(c) If k is constant but M varies, show that

$$z(t) = e^{-kt}\int_0^t \frac{ke^{ks}}{M(s)}\,ds + Ce^{-kt}$$

and use l'Hospital's Rule to deduce that if $M(t)$ has a limit as $t\to\infty$, then $P(t)$ has the same limit.

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Answers

1. Yes **3.** No **5.** $y = 1 + Ce^{-x}$

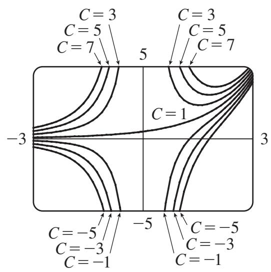
7. $y = x - 1 + Ce^{-x}$ **9.** $y = \frac{2}{3}\sqrt{x} + C/x$

11. $y = \dfrac{\int \sin(x^2)\,dx + C}{\sin x}$ **13.** $u = \dfrac{t^2 + 2t + 2C}{2(t + 1)}$

15. $y = \dfrac{1}{x}\ln x - \dfrac{1}{x} + \dfrac{3}{x^2}$ **17.** $u = -t^2 + t^3$

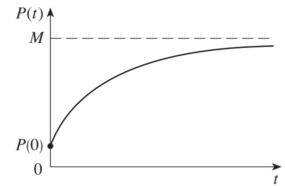
19. $y = -x\cos x - x$

21. $y = \dfrac{(x - 1)e^x + C}{x^2}$



25. $y = \pm\left(Cx^4 + \dfrac{2}{5x}\right)^{-1/2}$

27. $P(t) = M + Ce^{-kt}$



29. $y = \frac{2}{5}(100 + 2t) - 40{,}000(100 + 2t)^{-3/2}$; 0.2275 kg/L

31. (b) mg/c (c) $(mg/c)[t + (m/c)e^{-ct/m}] - m^2g/c^2$

33. (b) $P(t) = \dfrac{M}{1 + MCe^{-kt}}$

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Solutions

1. $x - y' = xy \quad \Leftrightarrow \quad y' + xy = x$ is linear since it can be put into the standard linear form (1), $y' + P(x)\,y = Q(x)$.

3. $y' = \dfrac{1}{x} + \dfrac{1}{y}$ is not linear since it cannot be put into the standard linear form (1), $y' + P(x)\,y = Q(x)$.

5. Comparing the given equation, $y' + y = 1$, with the general form, $y' + P(x)\,y = Q(x)$, we see that $P(x) = 1$ and the integrating factor is $I(x) = e^{\int P(x)\,dx} = e^{\int 1\,dx} = e^x$. Multiplying the differential equation by $I(x)$ gives

$e^x y' + e^x y = e^x \quad \Rightarrow \quad (e^x y)' = e^x \quad \Rightarrow \quad e^x y = \int e^x\,dx \quad \Rightarrow \quad e^x y = e^x + C \quad \Rightarrow \quad \dfrac{e^x y}{e^x} = \dfrac{e^x}{e^x} + \dfrac{C}{e^x} \quad \Rightarrow$

$y = 1 + Ce^{-x}$.

7. $y' = x - y \quad \Rightarrow \quad y' + y = x \ (\star)$. $I(x) = e^{\int P(x)\,dx} = e^{\int 1\,dx} = e^x$. Multiplying the differential equation $(\star)$ by $I(x)$ gives

$e^x y' + e^x y = xe^x \quad \Rightarrow \quad (e^x y)' = xe^x \quad \Rightarrow \quad e^x y = \int xe^x\,dx \quad \Rightarrow \quad e^x y = xe^x - e^x + C$ [by parts] $\Rightarrow$

$y = x - 1 + Ce^{-x}$ [divide by e^x].

9. Since $P(x)$ is the derivative of the coefficient of y' [$P(x) = 1$ and the coefficient is x], we can write the differential equation $xy' + y = \sqrt{x}$ in the easily integrable form $(xy)' = \sqrt{x} \quad \Rightarrow \quad xy = \frac{2}{3}x^{3/2} + C \quad \Rightarrow \quad y = \frac{2}{3}\sqrt{x} + C/x$.

11. $\sin x\,\dfrac{dy}{dx} + (\cos x)\,y = \sin(x^2) \quad \Rightarrow \quad [(\sin x)\,y]' = \sin(x^2) \quad \Rightarrow \quad (\sin x)\,y = \int \sin(x^2)\,dx \quad \Rightarrow \quad y = \dfrac{\int \sin(x^2)\,dx + C}{\sin x}$.

13. $(1+t)\,\dfrac{du}{dt} + u = 1 + t,\ t > 0$ [divide by $1+t$] $\Rightarrow \quad \dfrac{du}{dt} + \dfrac{1}{1+t}\,u = 1 \ (\star)$, which has the form $u' + P(t)\,u = Q(t)$. The integrating factor is $I(t) = e^{\int P(t)\,dt} = e^{\int [1/(1+t)]\,dt} = e^{\ln(1+t)} = 1 + t$.

Multiplying $(\star)$ by $I(t)$ gives us our original equation back. We rewrite it as $[(1+t)u]' = 1 + t$. Thus,

$(1+t)u = \int (1+t)\,dt = t + \frac{1}{2}t^2 + C \quad \Rightarrow \quad u = \dfrac{t + \frac{1}{2}t^2 + C}{1+t}$ or $u = \dfrac{t^2 + 2t + 2C}{2(t+1)}$.

15. $x^2 y' + 2xy = \ln x \quad \Rightarrow \quad (x^2 y)' = \ln x \quad \Rightarrow \quad x^2 y = \int \ln x\,dx \quad \Rightarrow \quad x^2 y = x\ln x - x + C$ [by parts]. Since $y(1) = 2$,

$1^2(2) = 1\ln 1 - 1 + C \quad \Rightarrow \quad 2 = -1 + C \quad \Rightarrow \quad C = 3$, so $x^2 y = x\ln x - x + 3$, or $y = \dfrac{1}{x}\ln x - \dfrac{1}{x} + \dfrac{3}{x^2}$.

17. $t\,\dfrac{du}{dt} = t^2 + 3u \quad \Rightarrow \quad u' - \dfrac{3}{t}u = t \ (\star)$. $I(t) = e^{\int -3/t\,dt} = e^{-3\ln|t|} = (e^{\ln|t|})^{-3} = t^{-3}$ $[t > 0]$ $= \dfrac{1}{t^3}$. Multiplying $(\star)$ by $I(t)$ gives $\dfrac{1}{t^3}\,u' - \dfrac{3}{t^4}u = \dfrac{1}{t^2} \quad \Rightarrow \quad \left(\dfrac{1}{t^3}u\right)' = \dfrac{1}{t^2} \quad \Rightarrow \quad \dfrac{1}{t^3}\,u = \displaystyle\int \dfrac{1}{t^2}\,dt \quad \Rightarrow \quad \dfrac{1}{t^3}\,u = -\dfrac{1}{t} + C$. Since $u(2) = 4$,

$\dfrac{1}{2^3}(4) = -\dfrac{1}{2} + C \quad \Rightarrow \quad C = 1$, so $\dfrac{1}{t^3}\,u = -\dfrac{1}{t} + 1$, or $u = -t^2 + t^3$.

19. $xy' = y + x^2 \sin x \quad \Rightarrow \quad y' - \dfrac{1}{x}\,y = x\sin x$. $I(x) = e^{\int (-1/x)\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = \dfrac{1}{x}$.

Multiplying by $\dfrac{1}{x}$ gives $\dfrac{1}{x}\,y' - \dfrac{1}{x^2}\,y = \sin x \quad \Rightarrow \quad \left(\dfrac{1}{x}\,y\right)' = \sin x \quad \Rightarrow \quad \dfrac{1}{x}\,y = -\cos x + C \quad \Rightarrow \quad y = -x\cos x + Cx$.

$y(\pi) = 0 \quad \Rightarrow \quad -\pi \cdot (-1) + C\pi = 0 \quad \Rightarrow \quad C = -1$, so $y = -x\cos x - x$.

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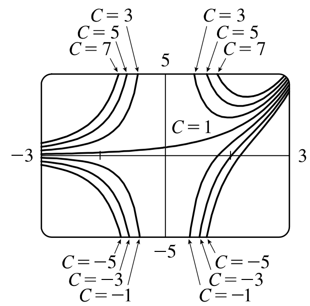
21. $xy' + 2y = e^x \quad \Rightarrow \quad y' + \dfrac{2}{x}\,y = \dfrac{e^x}{x}$.



$I(x) = e^{\int (2/x)\,dx} = e^{2\ln|x|} = \left(e^{\ln|x|}\right)^2 = |x|^2 = x^2$.

Multiplying by $I(x)$ gives $x^2\,y' + 2xy = xe^x \quad \Rightarrow \quad (x^2y)' = xe^x \quad \Rightarrow$

$x^2y = \int xe^x\,dx = (x-1)e^x + C \quad$ [by parts] $\quad \Rightarrow$

$y = [(x-1)e^x + C]/x^2$. The graphs for $C = -5, -3, -1, 1, 3, 5$, and 7 are shown. $C = 1$ is a transitional value. For $C < 1$, there is an inflection point and for $C > 1$, there is a local minimum. As $|C|$ gets larger, the "branches" get further from the origin.

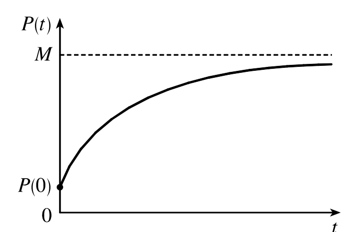
23. Setting $u = y^{1-n}$, $\dfrac{du}{dx} = (1-n)\,y^{-n}\dfrac{dy}{dx}$ or $\dfrac{dy}{dx} = \dfrac{y^n}{1-n}\dfrac{du}{dx} = \dfrac{u^{n/(1-n)}}{1-n}\dfrac{du}{dx}$. Then the Bernoulli differential equation becomes $\dfrac{u^{n/(1-n)}}{1-n}\dfrac{du}{dx} + P(x)u^{1/(1-n)} = Q(x)u^{n/(1-n)}$ or $\dfrac{du}{dx} + (1-n)P(x)u = Q(x)(1-n)$.

25. Here $y' + \dfrac{2}{x}\,y = \dfrac{y^3}{x^2}$, so $n = 3$, $P(x) = \dfrac{2}{x}$ and $Q(x) = \dfrac{1}{x^2}$. Setting $u = y^{-2}$, u satisfies $u' - \dfrac{4u}{x} = -\dfrac{2}{x^2}$.

Then $I(x) = e^{\int (-4/x)\,dx} = x^{-4}$ and $u = x^4\left(\displaystyle\int -\frac{2}{x^6}\,dx + C\right) = x^4\left(\dfrac{2}{5x^5} + C\right) = Cx^4 + \dfrac{2}{5x}$.

Thus, $y = \pm\left(Cx^4 + \dfrac{2}{5x}\right)^{-1/2}$.

27. $\dfrac{dP}{dt} + kP = kM$, so $I(t) = e^{\int k\,dt} = e^{kt}$. Multiplying the differential equation by $I(t)$ gives $e^{kt}\dfrac{dP}{dt} + kPe^{kt} = kMe^{kt} \quad \Rightarrow \quad (e^{kt}P)' = kMe^{kt} \quad \Rightarrow$



$P(t) = e^{-kt}\left(\int kMe^{kt}dt + C\right) = M + Ce^{-kt}$, $k > 0$. Furthermore, it is reasonable to assume that $0 \le P(0) \le M$, so $-M \le C \le 0$.

29. $y(0) = 0$ kg. Salt is added at a rate of $\left(0.4\,\dfrac{\text{kg}}{\text{L}}\right)\left(5\,\dfrac{\text{L}}{\text{min}}\right) = 2\,\dfrac{\text{kg}}{\text{min}}$. Since solution is drained from the tank at a rate of 3 L/min, but salt solution is added at a rate of 5 L/min, the tank, which starts out with 100 L of water, contains $(100 + 2t)$ L of liquid after t min. Thus, the salt concentration at time t is $\dfrac{y(t)}{100+2t}\,\dfrac{\text{kg}}{\text{L}}$. Salt therefore leaves the tank at a rate of $\left(\dfrac{y(t)}{100+2t}\,\dfrac{\text{kg}}{\text{L}}\right)\left(3\,\dfrac{\text{L}}{\text{min}}\right) = \dfrac{3y}{100+2t}\,\dfrac{\text{kg}}{\text{min}}$. Combining the rates at which salt enters and leaves the tank, we get $\dfrac{dy}{dt} = 2 - \dfrac{3y}{100+2t}$. Rewriting this equation as $\dfrac{dy}{dt} + \left(\dfrac{3}{100+2t}\right)y = 2$, we see that it is linear.

$$I(t) = \exp\left(\int \frac{3\,dt}{100+2t}\right) = \exp\left(\tfrac{3}{2}\ln(100+2t)\right) = (100+2t)^{3/2}$$

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Multiplying the differential equation by $I(t)$ gives $(100+2t)^{3/2}\dfrac{dy}{dt}+3(100+2t)^{1/2}y=2(100+2t)^{3/2} \quad\Rightarrow$

$[(100+2t)^{3/2}y]'=2(100+2t)^{3/2} \quad\Rightarrow\quad (100+2t)^{3/2}y=\frac{2}{5}(100+2t)^{5/2}+C \quad\Rightarrow$

$y=\frac{2}{5}(100+2t)+C(100+2t)^{-3/2}$. Now $0=y(0)=\frac{2}{5}(100)+C\cdot 100^{-3/2}=40+\frac{1}{1000}C \quad\Rightarrow\quad C=-40{,}000$, so

$y=\left[\frac{2}{5}(100+2t)-40{,}000(100+2t)^{-3/2}\right]$ kg. From this solution (no pun intended), we calculate the salt concentration

at time t to be $C(t)=\dfrac{y(t)}{100+2t}=\left[\dfrac{-40{,}000}{(100+2t)^{5/2}}+\dfrac{2}{5}\right]\dfrac{\text{kg}}{\text{L}}$. In particular, $C(20)=\dfrac{-40{,}000}{140^{5/2}}+\dfrac{2}{5}\approx 0.2275\,\dfrac{\text{kg}}{\text{L}}$

and $y(20)=\frac{2}{5}(140)-40{,}000(140)^{-3/2}\approx 31.85$ kg.

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31. (a) $\dfrac{dv}{dt}+\dfrac{c}{m}v=g$ and $I(t)=e^{\int (c/m)\,dt}=e^{(c/m)t}$, and multiplying the differential equation by

$I(t)$ gives $e^{(c/m)t}\dfrac{dv}{dt}+\dfrac{vce^{(c/m)t}}{m}=ge^{(c/m)t} \quad\Rightarrow\quad \left[e^{(c/m)t}v\right]'=ge^{(c/m)t}$. Hence,

$v(t)=e^{-(c/m)t}\left[\int ge^{(c/m)t}\,dt+K\right]=mg/c+Ke^{-(c/m)t}$. But the object is dropped from rest, so $v(0)=0$ and

$K=-mg/c$. Thus, the velocity at time t is $v(t)=(mg/c)[1-e^{-(c/m)t}]$.

(b) $\lim\limits_{t\to\infty} v(t)=mg/c$

(c) $s(t)=\int v(t)\,dt=(mg/c)[t+(m/c)e^{-(c/m)t}]+c_1$ where $c_1=s(0)-m^2g/c^2$.

$s(0)$ is the initial position, so $s(0)=0$ and $s(t)=(mg/c)[t+(m/c)e^{-(c/m)t}]-m^2g/c^2$.

33. (a) $z=\dfrac{1}{P} \quad\Rightarrow\quad P=\dfrac{1}{z} \quad\Rightarrow\quad P'=-\dfrac{z'}{z^2}$. Substituting into $P'=kP(1-P/M)$ gives us $-\dfrac{z'}{z^2}=k\,\dfrac{1}{z}\left(1-\dfrac{1}{zM}\right) \quad\Rightarrow$

$z'=-kz\left(1-\dfrac{1}{zM}\right) \quad\Rightarrow\quad z'=-kz+\dfrac{k}{M} \quad\Rightarrow\quad z'+kz=\dfrac{k}{M}$ $(\star)$.

(b) The integrating factor is $e^{\int k\,dt}=e^{kt}$. Multiplying $(\star)$ by e^{kt} gives $e^{kt}z'+ke^{kt}z=\dfrac{ke^{kt}}{M} \quad\Rightarrow\quad (e^{kt}z)'=\dfrac{k}{M}\,e^{kt} \quad\Rightarrow$

$e^{kt}z=\displaystyle\int \frac{k}{M}\,e^{kt}\,dt \quad\Rightarrow\quad e^{kt}z=\frac{1}{M}\,e^{kt}+C \quad\Rightarrow\quad z=\frac{1}{M}+Ce^{-kt}$. Since $P=\dfrac{1}{z}$, we have

$P=\dfrac{1}{\dfrac{1}{M}+Ce^{-kt}} \quad\Rightarrow\quad P=\dfrac{M}{1+MCe^{-kt}}$, which agrees with Equation 9.4.7, $P=\dfrac{M}{1+Ae^{-kt}}$, when $MC=A$.