2.1 DERIVATIVES AND RATES OF CHANGE

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EXAMPLE A The flash unit on a camera operates by storing charge on a capacitor and releasing it suddenly when the flash is set off. The data in the margin describe the charge Q remaining on the capacitor (measured in microcoulombs) at time t (measured in seconds after the flash goes off). Use the data to draw the graph of this function and estimate the slope of the tangent line at the point where $t = 0.04$. [*Note:* The slope of the tangent line represents the electric current flowing from the capacitor to the flash bulb (measured in microamperes).]

t	Q
0.00	100.00
0.02	81.87
0.04	67.03
0.06	54.88
0.08	44.93
0.10	36.76

SOLUTION In Figure 1 we plot the given data and use them to sketch a curve that approximates the graph of the function.

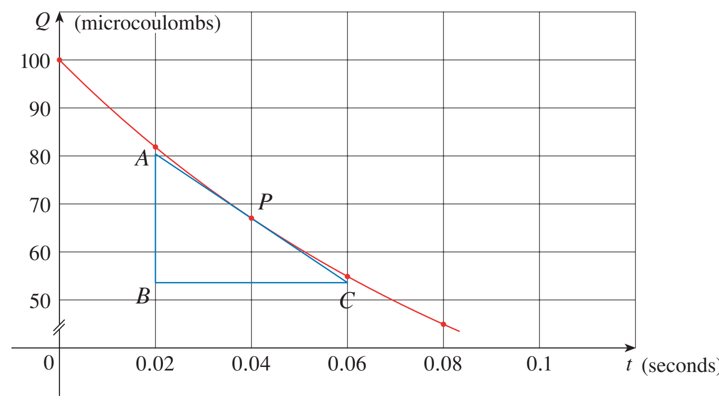

FIGURE 1

Given the points $P(0.04, 67.03)$ and $R(0.00, 100.00)$ on the graph, we find that the slope of the secant line PR is

$$m_{PR} = \frac{100.00 - 67.03}{0.00 - 0.04} = -824.25$$

R	m_{PR}
(0.00, 100.00)	−824.25
(0.02, 81.87)	−742.00
(0.06, 54.88)	−607.50
(0.08, 44.93)	−552.50
(0.10, 36.76)	−504.50

The table at the left shows the results of similar calculations for the slopes of other secant lines. From this table we would expect the slope of the tangent line at $t = 0.04$ to lie somewhere between -742 and -607.5. In fact, the average of the slopes of the two closest secant lines is

$$\tfrac{1}{2}(-742 - 607.5) = -674.75$$

So, by this method, we estimate the slope of the tangent line to be -675.

Another method is to draw an approximation to the tangent line at P and measure the sides of the triangle ABC, as in Figure 1. This gives an estimate of the slope of the tangent line as

$$-\frac{|AB|}{|BC|} \approx -\frac{80.4 - 53.6}{0.06 - 0.02} = -670$$

■ The physical meaning of the answer in Example A is that the electric current flowing from the capacitor to the flash bulb after 0.04 second is about -670 microamperes.

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x (h)	T (°C)	x (h)	T (°C)
0	6.5	13	16.0
1	6.1	14	17.3
2	5.6	15	18.2
3	4.9	16	18.8
4	4.2	17	17.6
5	4.0	18	16.0
6	4.0	19	14.1
7	4.8	20	11.5
8	6.1	21	10.2
9	8.3	22	9.0
10	10.0	23	7.9
11	12.1	24	7.0
12	14.3		

EXAMPLE B Temperature readings T (in degrees Celsius) were recorded every hour starting at midnight on a day in April in Whitefish, Montana. The time x is measured in hours from midnight. The data are given in the table at the left.

(a) Find the average rate of change of temperature with respect to time

(i) from noon to 3 P.M. (ii) from noon to 2 P.M.

(iii) from noon to 1 P.M.

(b) Estimate the instantaneous rate of change at noon.

SOLUTION

(a) (i) From noon to 3 P.M. the temperature changes from 14.3°C to 18.2°C, so

$$\Delta T = T(15) - T(12) = 18.2 - 14.3 = 3.9°\text{C}$$

while the change in time is $\Delta x = 3$ h. Therefore, the average rate of change of temperature with respect to time is

$$\frac{\Delta T}{\Delta x} = \frac{3.9}{3} = 1.3°\text{C/h}$$

(ii) From noon to 2 P.M. the average rate of change is

$$\frac{\Delta T}{\Delta x} = \frac{T(14) - T(12)}{14 - 12} = \frac{17.3 - 14.3}{2} = 1.5°\text{C/h}$$

(iii) From noon to 1 P.M. the average rate of change is

$$\frac{\Delta T}{\Delta x} = \frac{T(13) - T(12)}{13 - 12} = \frac{16.0 - 14.3}{1} = 1.7°\text{C/h}$$

▪ **A NOTE ON UNITS**

The units for the average rate of change $\Delta T/\Delta x$ are the units for ΔT divided by the units for Δx, namely, degrees Celsius per hour. The instantaneous rate of change is the limit of the average rates of change, so it is measured in the same units: degrees Celsius per hour.

(b) We plot the given data in Figure 2 and use them to sketch a smooth curve that approximates the graph of the temperature function. Then we draw the tangent at the point P where $x = 12$ and, after measuring the sides of triangle ABC, we estimate that the slope of the tangent line is

$$\frac{|BC|}{|AC|} = \frac{10.3}{5.5} \approx 1.9$$

▪ Another method is to average the slopes of two secant lines. See Example A.

Therefore, the instantaneous rate of change of temperature with respect to time at noon is about 1.9°C/h.

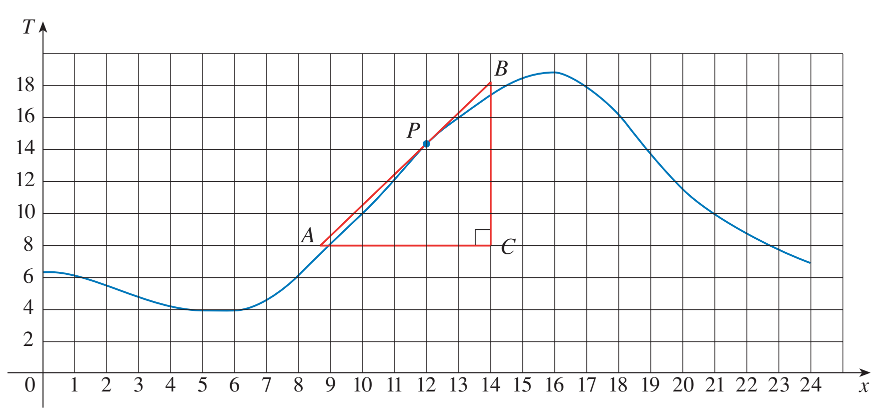

FIGURE 2

EXAMPLE C The position of a particle is given by the equation of motion $s = f(t) = 1/(1 + t)$, where t is measured in seconds and s in meters. Find the velocity and the speed after 2 seconds.

SOLUTION The derivative of f when $t = 2$ is

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{1+(2+h)} - \dfrac{1}{1+2}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{3+h} - \dfrac{1}{3}}{h} = \lim_{h\to 0} \frac{\dfrac{3-(3+h)}{3(3+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-h}{3(3+h)h} = \lim_{h\to 0} \frac{-1}{3(3+h)} = -\frac{1}{9}$$

Thus, the velocity after 2 seconds is $f'(2) = -\frac{1}{9}$ m/s, and the speed is $|f'(2)| = |-\frac{1}{9}| = \frac{1}{9}$ m/s. ■

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V EXAMPLE D A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is $C = f(x)$ dollars.

(a) What is the meaning of the derivative $f'(x)$? What are its units?
(b) In practical terms, what does it mean to say that $f'(1000) = 9$?
(c) Which do you think is greater, $f'(50)$ or $f'(500)$? What about $f'(5000)$?

SOLUTION

(a) The derivative $f'(x)$ is the instantaneous rate of change of C with respect to x; that is, $f'(x)$ means the rate of change of the production cost with respect to the number of yards produced. (Economists call this rate of change the *marginal cost.* This idea is discussed in more detail in Sections 2.3 and 4.5.)

Because

$$f'(x) = \lim_{\Delta x\to 0} \frac{\Delta C}{\Delta x}$$

the units for $f'(x)$ are the same as the units for the difference quotient $\Delta C/\Delta x$. Since ΔC is measured in dollars and Δx in yards, it follows that the units for $f'(x)$ are dollars per yard.

(b) The statement that $f'(1000) = 9$ means that, after 1000 yards of fabric have been manufactured, the rate at which the production cost is increasing is \$9/yard. (When $x = 1000$, C is increasing 9 times as fast as x.)
Since $\Delta x = 1$ is small compared with $x = 1000$, we could use the approximation

▪ Here we are assuming that the cost function is well behaved; in other words, $C(x)$ doesn't oscillate rapidly near $x = 1000$.

$$f'(1000) \approx \frac{\Delta C}{\Delta x} = \frac{\Delta C}{1} = \Delta C$$

and say that the cost of manufacturing the 1000th yard (or the 1001st) is about \$9.

(c) The rate at which the production cost is increasing (per yard) is probably lower when $x = 500$ than when $x = 50$ (the cost of making the 500th yard is less than the cost of the 50th yard) because of economies of scale. (The manufacturer makes more efficient use of the fixed costs of production.) So

$$f'(50) > f'(500)$$

But, as production expands, the resulting large-scale operation might become inefficient and there might be overtime costs. Thus, it is possible that the rate of increase of costs will eventually start to rise. So it may happen that

$$f'(5000) > f'(500)$$ ■