Formulas for the Remainder Term in Taylor Series

In Section 11.10 we considered functions f with derivatives of all orders and their **Taylor series**

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

The nth partial sum of this Taylor series is the ***n*th-degree Taylor polynomial of *f* at *a***:

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

We can write

$$f(x) = T_n(x) + R_n(x)$$

where $R_n(x)$ is the **remainder** of the Taylor series. We know that f is equal to the sum of its Taylor series on the interval $|x - a| < R$ if we can show that $\lim_{n\to\infty} R_n(x) = 0$ for $|x - a| < R$.

Here we derive formulas for the remainder term $R_n(x)$. The first such formula involves an integral.

1 Theorem If $f^{(n+1)}$ is continuous on an open interval I that contains a, and x is in I, then

$$R_n(x) = \frac{1}{n!} \int_a^x (x - t)^n f^{(n+1)}(t)\, dt$$

Proof We use mathematical induction. For $n = 1$,

$$R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f'(a)(x - a)$$

and the integral in the theorem is $\int_a^x (x - t) f''(t)\, dt$. To evaluate this integral we integrate by parts with $u = x - t$ and $dv = f''(t)\, dt$, so $du = -dt$ and $v = f'(t)$. Thus

$$\begin{aligned}
\int_a^x (x - t) f''(t)\, dt &= (x - t) f'(t)\Big]_{t=a}^{t=x} + \int_a^x f'(t)\, dt \\
&= 0 - (x - a) f'(a) + f(x) - f(a) \qquad \text{(by FTC 2)} \\
&= f(x) - f(a) - f'(a)(x - a) = R_1(x)
\end{aligned}$$

The theorem is therefore proved for $n = 1$.

Now we suppose that Theorem 1 is true for $n = k$, that is,

$$R_k(x) = \frac{1}{k!} \int_a^x (x - t)^k f^{(k+1)}(t)\, dt$$

We want to show that it's true for $n = k + 1$, that is

$$R_{k+1}(x) = \frac{1}{(k + 1)!} \int_a^x (x - t)^{k+1} f^{(k+2)}(t)\, dt$$

Again we use integration by parts, this time with $u = (x - t)^{k+1}$ and $dv = f^{(k+2)}(t)$. Then $du = -(k + 1)(x - t)^k \, dt$ and $v = f^{(k+1)}(t)$, so

$$\begin{aligned}
\frac{1}{(k+1)!} & \int_a^x (x - t)^{k+1} f^{(k+2)}(t) \, dt \\
&= \frac{1}{(k+1)!} (x - t)^{k+1} f^{(k+1)}(t) \Big]_{t=a}^{t=x} + \frac{k+1}{(k+1)!} \int_a^x (x - t)^k f^{(k+1)}(t) \, dt \\
&= 0 - \frac{1}{(k+1)!} (x - a)^{k+1} f^{(k+1)}(a) + \frac{1}{k!} \int_a^x (x - t)^k f^{(k+1)}(t) \, dt \\
&= -\frac{f^{(k+1)}(a)}{(k+1)!} (x - a)^{k+1} + R_k(x) \\
&= f(x) - T_k(x) - \frac{f^{(k+1)}(a)}{(k+1)!} (x - a)^{k+1} \\
&= f(x) - T_{k+1}(x) = R_{k+1}(x)
\end{aligned}$$

Therefore, (1) is true for $n = k + 1$ when it is true for $n = k$. Thus, by mathematical induction, it is true for all n.

To illustrate Theorem 1 we use it to solve Example 4 in Section 11.10.

EXAMPLE 1 Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

SOLUTION We arrange our computation in two columns as follows:

$$\begin{aligned}
f(x) &= \sin x & f(0) &= 0 \\
f'(x) &= \cos x & f'(0) &= 1 \\
f''(x) &= -\sin x & f''(0) &= 0 \\
f'''(x) &= -\cos x & f'''(0) &= -1 \\
f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0
\end{aligned}$$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$\begin{aligned}
f(0) &+ \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \cdots \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

With $a = 0$ in Theorem 1, we have

$$R_n(x) = \frac{1}{n!} \int_0^x (x - t)^n f^{(n+1)}(t) \, dt$$

Since $f^{(n+1)}(t)$ is $\pm\sin t$ or $\pm\cos t$, we know that $|f^{(n+1)}(t)| \leqslant 1$ for all t. We use the fact (see Exercise 65 in Section 5.2) that, for $a \leqslant b$,

$$\left| \int_a^b f(t)\, dt \right| \leqslant \int_a^b |f(t)|\, dt$$

Thus, for $x > 0$,

$$|R_n(x)| = \frac{1}{n!}\left| \int_0^x (x-t)^n f^{(n+1)}(t)\, dt \right| \leqslant \frac{1}{n!}\int_0^x (x-t)^n |f^{(n+1)}(t)|\, dt$$

$$\leqslant \frac{1}{n!}\int_0^x (x-t)^n\, dt = \frac{1}{n!}\,\frac{x^{n+1}}{n+1} = \frac{x^{n+1}}{(n+1)!}$$

For $x < 0$ we can write

$$R_n(x) = -\frac{1}{n!}\int_x^0 (x-t)^n f^{(n+1)}(t)\, dt$$

so

$$|R_n(x)| \leqslant \frac{1}{n!}\left| \int_x^0 |x-t|^n\, |f^{(n+1)}(t)|\, dt \leqslant \frac{1}{n!}\int_x^0 (t-x)^n\, dt = \frac{(-x)^{n+1}}{(n+1)!}$$

Thus, in any case, we have

$$|R_n(x)| \leqslant \frac{|x|^{n+1}}{(n+1)!}$$

The right side of this inequality approaches 0 as $n \to \infty$ (see Equation 11.10.10), so $|R_n(x)| \to 0$ by the Squeeze Theorem. It follows that $R_n(x) \to 0$ as $n \to \infty$, so $\sin x$ is equal to the sum of its Maclaurin series.

For some purposes the integral formula in Theorem 1 is awkward to work with, so we are going to establish another formula for the remainder term. To that end we need to prove the following generalization of the Mean Value Theorem for Integrals (see Section 6.5).

2 Weighted Mean Value Theorem for Integrals If f and g are continuous on $[a, b]$ and g does not change sign in $[a, b]$, then there exists a number c in $[a, b]$ such that

$$\int_a^b f(x)g(x)\, dx = f(c)\int_a^b g(x)\, dx$$

Proof Because g doesn't change sign, either $g(x) \geqslant 0$ or $g(x) \leqslant 0$ for $a \leqslant x \leqslant b$. For the sake of definiteness, let's assume that $g(x) \geqslant 0$.

By the Extreme Value Theorem (4.1.3), f has an absolute minimum value m and an absolute maximum value M, so $m \leqslant f(x) \leqslant M$ for $a \leqslant x \leqslant b$. Since $g(x) \geqslant 0$, we have

$$mg(x) \leqslant f(x)g(x) \leqslant Mg(x) \qquad a \leqslant x \leqslant b$$

and so

$$\boxed{3} \qquad m \int_a^b g(x)\,dx \leqslant \int_a^b f(x)g(x)\,dx \leqslant M \int_a^b g(x)\,dx$$

If $\int_a^b g(x)\,dx = 0$, these inequalities show that $\int_a^b f(x)g(x)\,dx = 0$ and so Theorem 2 is true because both sides of the equation are 0. If $\int_a^b g(x)\,dx \neq 0$, it must be positive and we can divide by $\int_a^b g(x)\,dx$ in (3):

$$m \leqslant \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \leqslant M$$

Then, by the Intermediate Value Theorem (2.5.10), there exists a number c in $[a, b]$ such that

$$f(c) = \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \qquad \text{and so} \qquad \int_a^b f(x)g(x)\,dx = f(c) \int_a^b g(x)\,dx$$

$\boxed{4}$ **Theorem** If $f^{(n+1)}$ is continuous on an open interval I that contains a, and x is in I, then there exists a number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Proof The function $g(t) = (x-t)^n$ doesn't change sign in the interval from a to x, so the Weighted Mean Value Theorem for Integrals gives a number c between a and x such that

$$\int_a^x (x-t)^n f^{(n+1)}(t)\,dt = f^{(n+1)}(c) \int_a^x (x-t)^n\,dt$$

$$= -f^{(n+1)}(c) \frac{(x-t)^{n+1}}{n+1}\bigg]_{t=a}^{t=x} = f^{(n+1)}(c) \frac{(x-a)^{n+1}}{n+1}$$

Then, by Theorem 1,

$$R_n(x) = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

$$= \frac{1}{n!} f^{(n+1)}(c) \frac{(x-a)^{n+1}}{n+1} = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

The formula for the remainder term in Theorem 4 is called **Lagrange's form of the remainder term**. Notice that this expression

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

is very similar to the terms in the Taylor series except that $f^{(n+1)}$ is evaluated at c instead of at a. All we can say about the number c is that it lies somewhere between x and a.

In the following example we show how to use Lagrange's form of the remainder term as an alternative to the integral form in Example 1.

EXAMPLE 2 Prove that Maclaurin series for $\sin x$ represents $\sin x$ for all x.

SOLUTION Using the Lagrange form of the remainder term with $a = 0$, we have

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

where $f(x) = \sin x$ and c lies between 0 and x. But $f^{(n+1)}(c)$ is $\pm\sin c$ or $\pm\cos c$. In any case, $|f^{(n+1)}(c)| \leqslant 1$ and so

$$|R_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} |x^{n+1}| \leqslant \frac{|x|^{n+1}}{(n+1)!}$$

By Equation 11.10.10 the right side of this inequality approaches 0 as $n \to \infty$, so $|R_n(x)| \to 0$ by the Squeeze Theorem. It follows that $R_n(x) \to 0$ as $n \to \infty$, so $\sin x$ is equal to the sum of its Maclaurin series.

EXAMPLE 3

(a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.
(b) How accurate is this approximation when $7 \leqslant x \leqslant 9$?

SOLUTION

(a)
$$\begin{aligned} f(x) &= \sqrt[3]{x} = x^{1/3} & f(8) &= 2 \\ f'(x) &= \tfrac{1}{3}x^{-2/3} & f'(8) &= \tfrac{1}{12} \\ f''(x) &= -\tfrac{2}{9}x^{-5/3} & f''(8) &= -\tfrac{1}{144} \\ f'''(x) &= \tfrac{10}{27}x^{-8/3} \end{aligned}$$

Thus the second-degree Taylor polynomial is

$$\begin{aligned} T_2(x) &= f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2 \end{aligned}$$

The desired approximation is

$$\sqrt[3]{x} \approx T_2(x) = 2 + \tfrac{1}{12}(x-8) - \tfrac{1}{288}(x-8)^2$$

(b) Using the Lagrange form of the remainder term we can write

$$R_2(x) = \frac{f'''(c)}{3!}(x-8)^3 = \tfrac{10}{27}c^{-8/3}\frac{(x-8)^3}{3!} = \frac{5(x-8)^3}{81c^{8/3}}$$

where c lies between 8 and x. In order to estimate the error we note that if $7 \leqslant x \leqslant 9$, then $-1 \leqslant x - 8 \leqslant 1$, so $|x-8| \leqslant 1$ and therefore $|x-8|^3 \leqslant 1$. Also, since $x > 7$, we have

$$c^{8/3} > 7^{8/3} > 179$$

and so

$$|R_2(x)| = \frac{5|x-8|^3}{81c^{8/3}} < \frac{5 \cdot 1}{81 \cdot 179} < 0.0004$$

Thus if $7 \leqslant x \leqslant 9$, the approximation in part (a) is accurate to within 0.0004.