Rotation of Axes

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▲ For a discussion of conic sections, see *Calculus*, Fourth Edition, Section 11.6 *Calculus, Early Transcendentals,* Fourth Edition, Section 10.6

In precalculus or calculus you may have studied conic sections with equations of the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

Here we show that the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{1}$$

can be analyzed by rotating the axes so as to eliminate the term Bxy.

In Figure 1 the x and y axes have been rotated about the origin through an acute angle θ to produce the X and Y axes. Thus, a given point P has coordinates (x, y) in the first coordinate system and (X, Y) in the new coordinate system. To see how X and Y are related to x and y we observe from Figure 2 that

$$X = r\cos\phi \qquad Y = r\sin\phi$$

$$x = r\cos(\theta + \phi) \qquad y = r\sin(\theta + \phi)$$

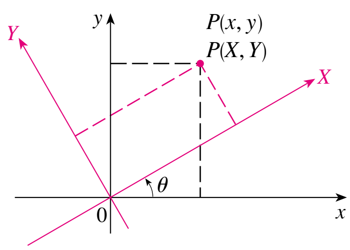


FIGURE 1

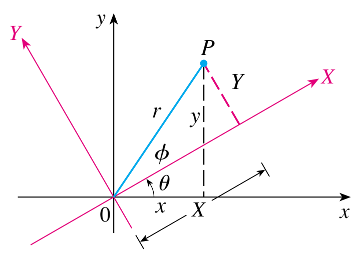


FIGURE 2

The addition formula for the cosine function then gives

$$x = r\cos(\theta + \phi) = r(\cos\theta\cos\phi - \sin\theta\sin\phi)$$

$$= (r\cos\phi)\cos\theta - (r\sin\phi)\sin\theta = X\cos\theta - Y\sin\theta$$

A similar computation gives y in terms of X and Y and so we have the following formulas:

$$x = X\cos\theta - Y\sin\theta \qquad y = X\sin\theta + Y\cos\theta \tag{2}$$

By solving Equations 2 for X and Y we obtain

$$X = x\cos\theta + y\sin\theta \qquad Y = -x\sin\theta + y\cos\theta \tag{3}$$

EXAMPLE 1 If the axes are rotated through 60°, find the XY-coordinates of the point whose xy-coordinates are (2, 6).

SOLUTION Using Equations 3 with $x = 2$, $y = 6$, and $\theta = 60°$, we have

$$X = 2\cos 60° + 6\sin 60° = 1 + 3\sqrt{3}$$

$$Y = -2\sin 60° + 6\cos 60° = -\sqrt{3} + 3$$

The XY-coordinates are $(1 + 3\sqrt{3}, 3 - \sqrt{3})$.

Now let's try to determine an angle θ such that the term Bxy in Equation 1 disappears when the axes are rotated through the angle θ. If we substitute from Equations 2 in Equation 1, we get

$$A(X\cos\theta - Y\sin\theta)^2 + B(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta)$$

$$+ C(X\sin\theta + Y\cos\theta)^2 + D(X\cos\theta - Y\sin\theta)$$

$$+ E(X\sin\theta + Y\cos\theta) + F = 0$$

Expanding and collecting terms, we obtain an equation of the form

4 $$A'X^2 + B'XY + C'Y^2 + D'X + E'Y + F = 0$$

where the coefficient B' of XY is

$$B' = 2(C - A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta)$$

$$= (C - A)\sin 2\theta + B\cos 2\theta$$

To eliminate the XY term we choose θ so that $B' = 0$, that is,

$$(A - C)\sin 2\theta = B\cos 2\theta$$

or

5 $$\cot 2\theta = \frac{A - C}{B}$$

EXAMPLE 2 Show that the graph of the equation $xy = 1$ is a hyperbola.

SOLUTION Notice that the equation $xy = 1$ is in the form of Equation 1 where $A = 0$, $B = 1$, and $C = 0$. According to Equation 5, the xy term will be eliminated if we choose θ so that

$$\cot 2\theta = \frac{A - C}{B} = 0$$

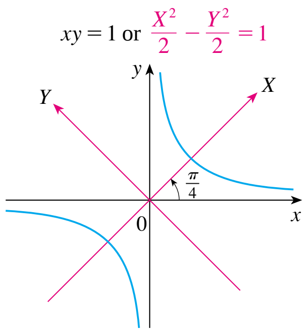


FIGURE 3

This will be true if $2\theta = \pi/2$, that is, $\theta = \pi/4$. Then $\cos\theta = \sin\theta = 1/\sqrt{2}$ and Equations 2 become

$$x = \frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}} \qquad y = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$$

Substituting these expressions into the original equation gives

$$\left(\frac{X}{\sqrt{2}} - \frac{Y}{\sqrt{2}}\right)\left(\frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}\right) = 1 \quad \text{or} \quad \frac{X^2}{2} - \frac{Y^2}{2} = 1$$

We recognize this as a hyperbola with vertices $(\pm\sqrt{2}, 0)$ in the XY-coordinate system. The asymptotes are $Y = \pm X$ in the XY-system, which correspond to the coordinate axes in the xy-system (see Figure 3). ■

EXAMPLE 3 Identify and sketch the curve

$$73x^2 + 72xy + 52y^2 + 30x - 40y - 75 = 0$$

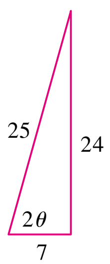


FIGURE 4

SOLUTION This equation is in the form of Equation 1 with $A = 73$, $B = 72$, and $C = 52$. Thus

$$\cot 2\theta = \frac{A - C}{B} = \frac{73 - 52}{72} = \frac{7}{24}$$

From the triangle in Figure 4 we see that

$$\cos 2\theta = \tfrac{7}{25}$$

The values of $\cos\theta$ and $\sin\theta$ can then be computed from the half-angle formulas:

$$\cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}} = \sqrt{\frac{1 + \frac{7}{25}}{2}} = \frac{4}{5}$$

$$\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}} = \sqrt{\frac{1 - \frac{7}{25}}{2}} = \frac{3}{5}$$

The rotation equations (2) become

$$x = \tfrac{4}{5}X - \tfrac{3}{5}Y \qquad y = \tfrac{3}{5}X + \tfrac{4}{5}Y$$

Substituting into the given equation, we have

$$73\left(\tfrac{4}{5}X - \tfrac{3}{5}Y\right)^2 + 72\left(\tfrac{4}{5}X - \tfrac{3}{5}Y\right)\left(\tfrac{3}{5}X + \tfrac{4}{5}Y\right) + 52\left(\tfrac{3}{5}X + \tfrac{4}{5}Y\right)^2$$
$$+ 30\left(\tfrac{4}{5}X - \tfrac{3}{5}Y\right) - 40\left(\tfrac{3}{5}X + \tfrac{4}{5}Y\right) - 75 = 0$$

which simplifies to $\qquad 4X^2 + Y^2 - 2Y = 3$

Completing the square gives

$$4X^2 + (Y - 1)^2 = 4 \qquad \text{or} \qquad X^2 + \frac{(Y - 1)^2}{4} = 1$$

and we recognize this as being an ellipse whose center is $(0, 1)$ in XY-coordinates.

Since $\theta = \cos^{-1}\left(\frac{4}{5}\right) \approx 37°$, we can sketch the graph in Figure 5.

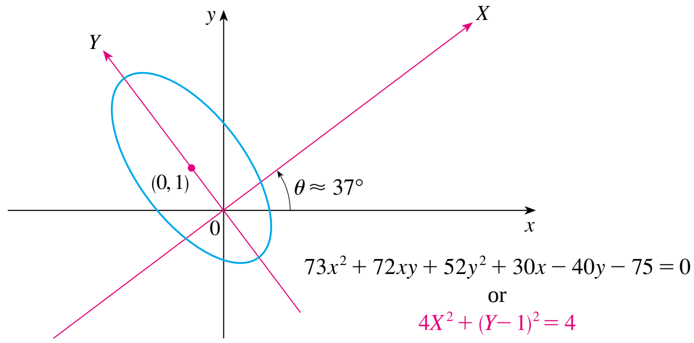


FIGURE 5

Exercises

A Click here for answers.

1–4 ■ Find the XY-coordinates of the given point if the axes are rotated through the specified angle.

1. $(1, 4)$, $30°$

2. $(4, 3)$, $45°$

3. $(-2, 4)$, $60°$

4. $(1, 1)$, $15°$

5–12 ■ Use rotation of axes to identify and sketch the curve.

5. $x^2 - 2xy + y^2 - x - y = 0$

6. $x^2 - xy + y^2 = 1$

7. $x^2 + xy + y^2 = 1$

8. $\sqrt{3}xy + y^2 = 1$

9. $97x^2 + 192xy + 153y^2 = 225$

10. $3x^2 - 12\sqrt{5}xy + 6y^2 + 9 = 0$

11. $2\sqrt{3}xy - 2y^2 - \sqrt{3}x - y = 0$

12. $16x^2 - 8\sqrt{2}xy + 2y^2 + \left(8\sqrt{2} - 3\right)x - \left(6\sqrt{2} + 4\right)y = 7$

13. (a) Use rotation of axes to show that the equation

$$36x^2 + 96xy + 64y^2 + 20x - 15y + 25 = 0$$

represents a parabola.

(b) Find the XY-coordinates of the focus. Then find the xy-coordinates of the focus.

(c) Find an equation of the directrix in the xy-coordinate system.

14. (a) Use rotation of axes to show that the equation

$$2x^2 - 72xy + 23y^2 - 80x - 60y = 125$$

represents a hyperbola.

(b) Find the XY-coordinates of the foci. Then find the xy-coordinates of the foci.

(c) Find the xy-coordinates of the vertices.

(d) Find the equations of the asymptotes in the xy-coordinate system.

(e) Find the eccentricity of the hyperbola.

15. Suppose that a rotation changes Equation 1 into Equation 4. Show that

$$A' + C' = A + C$$

16. Suppose that a rotation changes Equation 1 into Equation 4. Show that

$$(B')^2 - 4A'C' = B^2 - 4AC$$

17. Use Exercise 16 to show that Equation 1 represents (a) a parabola if $B^2 - 4AC = 0$, (b) an ellipse if $B^2 - 4AC < 0$, and (c) a hyperbola if $B^2 - 4AC > 0$, except in degenerate cases when it reduces to a point, a line, a pair of lines, or no graph at all.

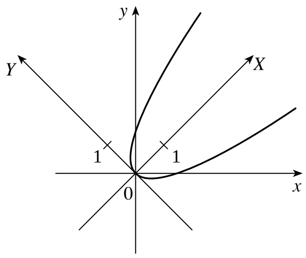
18. Use Exercise 17 to determine the type of curve in Exercises 9–12.

Answers

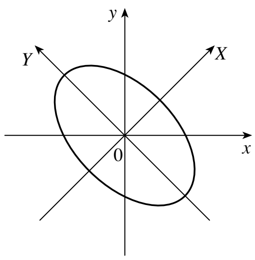
1. $\left((\sqrt{3}+4)/2, (4\sqrt{3}-1)/2\right)$

3. $\left(2\sqrt{3}-1, \sqrt{3}+2\right)$

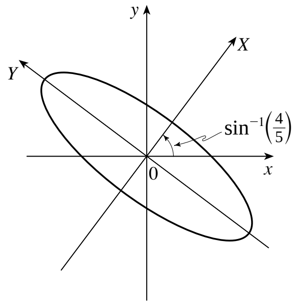
5. $X = \sqrt{2}\,Y^2$, parabola



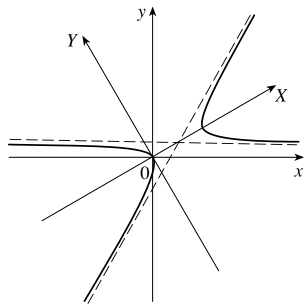
7. $3X^2 + Y^2 = 2$, ellipse



9. $X^2 + (Y^2/9) = 1$, ellipse



11. $(X-1)^2 - 3Y^2 = 1$, hyperbola



13. (a) $Y - 1 = 4X^2$ (b) $\left(0, \frac{17}{16}\right), \left(-\frac{17}{20}, \frac{51}{80}\right)$
(c) $64x - 48y + 75 = 0$