

CHALLENGE PROBLEMS

CHAPTER 5

A [Click here for answers.](#)

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1. Show that

$$\frac{d^n}{dx^n} (e^{ax} \sin bx) = r^n e^{ax} \sin(bx + n\theta)$$

where a and b are positive numbers, $r^2 = a^2 + b^2$, and $\theta = \tan^{-1}(b/a)$.

2. Show that $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$.

3. If

$$y = \frac{x}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}} \arctan \frac{\sin x}{a + \sqrt{a^2 - 1} + \cos x}$$

show that $y' = \frac{1}{a + \cos x}$.

4. For which positive numbers a is it true that $a^x \geq 1 + x$ for all x ?
5. For what value of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution?
6. A peach pie is taken out of the oven at 5:00 P.M. At that time it is piping hot: 100°C. At 5:10 P.M. its temperature is 80°C; at 5:20 P.M. it is 65°C. What is the temperature of the room?

ANSWERS

S Solutions

5. $2\sqrt{e}$

SOLUTIONS
E Exercises

1. Consider the statement that $\frac{d^n}{dx^n}(e^{ax} \sin bx) = r^n e^{ax} \sin(bx + n\theta)$. For $n = 1$,

$$\frac{d}{dx}(e^{ax} \sin bx) = ae^{ax} \sin bx + be^{ax} \cos bx, \text{ and}$$

$$\begin{aligned} re^{ax} \sin(bx + \theta) &= re^{ax} [\sin bx \cos \theta + \cos bx \sin \theta] = re^{ax} \left(\frac{a}{r} \sin bx + \frac{b}{r} \cos bx \right) \\ &= ae^{ax} \sin bx + be^{ax} \cos bx \end{aligned}$$

$$\text{since } \tan \theta = \frac{b}{a} \Rightarrow \sin \theta = \frac{b}{r} \text{ and } \cos \theta = \frac{a}{r}.$$

So the statement is true for $n = 1$. Assume it is true for $n = k$. Then

$$\begin{aligned} \frac{d^{k+1}}{dx^{k+1}}(e^{ax} \sin bx) &= \frac{d}{dx} [r^k e^{ax} \sin(bx + k\theta)] = r^k ae^{ax} \sin(bx + k\theta) + r^k e^{ax} b \cos(bx + k\theta) \\ &= r^k e^{ax} [a \sin(bx + k\theta) + b \cos(bx + k\theta)] \end{aligned}$$

But

$$\begin{aligned} \sin[bx + (k+1)\theta] &= \sin[(bx + k\theta) + \theta] = \sin(bx + k\theta) \cos \theta + \sin \theta \cos(bx + k\theta) \\ &= \frac{a}{r} \sin(bx + k\theta) + \frac{b}{r} \cos(bx + k\theta) \end{aligned}$$

Hence, $a \sin(bx + k\theta) + b \cos(bx + k\theta) = r \sin[bx + (k+1)\theta]$. So

$$\begin{aligned} \frac{d^{k+1}}{dx^{k+1}}(e^{ax} \sin bx) &= r^k e^{ax} [a \sin(bx + k\theta) + b \cos(bx + k\theta)] = r^k e^{ax} [r \sin(bx + (k+1)\theta)] \\ &= r^{k+1} e^{ax} [\sin(bx + (k+1)\theta)] \end{aligned}$$

Therefore, the statement is true for all n by mathematical induction.

3. $y = \frac{x}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}} \arctan \frac{\sin x}{a + \sqrt{a^2 - 1} + \cos x}$. Let $k = a + \sqrt{a^2 - 1}$. Then

$$\begin{aligned} y' &= \frac{1}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}} \cdot \frac{1}{1 + \sin^2 x / (k + \cos x)^2} \cdot \frac{\cos x(k + \cos x) + \sin^2 x}{(k + \cos x)^2} \\ &= \frac{1}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}} \cdot \frac{k \cos x + \cos^2 x + \sin^2 x}{(k + \cos x)^2 + \sin^2 x} = \frac{1}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}} \cdot \frac{k \cos x + 1}{k^2 + 2k \cos x + 1} \\ &= \frac{k^2 + 2k \cos x + 1 - 2k \cos x - 2}{\sqrt{a^2 - 1}(k^2 + 2k \cos x + 1)} = \frac{k^2 - 1}{\sqrt{a^2 - 1}(k^2 + 2k \cos x + 1)} \end{aligned}$$

But $k^2 = 2a^2 + 2a\sqrt{a^2 - 1} - 1 = 2a(a + \sqrt{a^2 - 1}) - 1 = 2ak - 1$, so $k^2 + 1 = 2ak$, and

$$k^2 - 1 = 2(ak - 1). \text{ So } y' = \frac{2(ak - 1)}{\sqrt{a^2 - 1}(2ak + 2k \cos x)} = \frac{ak - 1}{\sqrt{a^2 - 1}k(a + \cos x)}. \text{ But}$$

$$ak - 1 = a^2 + a\sqrt{a^2 - 1} - 1 = k\sqrt{a^2 - 1}, \text{ so } y' = 1/(a + \cos x).$$

5. Let $f(x) = e^{2x}$ and $g(x) = k\sqrt{x}$ ($k > 0$). From the graphs of f and g , we see that f will intersect g exactly once when f and g share a tangent line. Thus, we must have $f = g$ and $f' = g'$ at $x = a$.

$$f(a) = g(a) \Rightarrow e^{2a} = k\sqrt{a} \quad \mathbf{(1)} \quad \text{and} \quad f'(a) = g'(a) \Rightarrow$$

$$2e^{2a} = \frac{k}{2\sqrt{a}} \Rightarrow e^{2a} = \frac{k}{4\sqrt{a}}. \quad \text{So we must have } k\sqrt{a} = \frac{k}{4\sqrt{a}}$$

$$\Rightarrow (\sqrt{a})^2 = \frac{k}{4k} \Rightarrow a = \frac{1}{4}. \quad \text{From } \mathbf{(1)}, e^{2(1/4)} = k\sqrt{1/4} \Rightarrow k = 2e^{1/2} = 2\sqrt{e} \approx 3.297.$$

