8.5 POWER SERIES

A Click here for answers.

S Click here for solutions.

1–19 ▪ Find the radius of convergence and interval of convergence of the series.

1. $\sum_{n=0}^{\infty} \frac{x^n}{n+2}$

2. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n 2^n}$

4. $\sum_{n=1}^{\infty} n 5^n x^n$

5. $\sum_{n=0}^{\infty} n x^n$

6. $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

7. $\sum_{n=0}^{\infty} \frac{3^n x^n}{(n+1)^2}$

8. $\sum_{n=0}^{\infty} \frac{n^2 x^n}{10^n}$

9. $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$

10. $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!}$

11. $\sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{n+3}$

12. $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n(n+1)}$

13. $\sum_{n=0}^{\infty} \sqrt{n}(3x+2)^n$

14. $\sum_{n=0}^{\infty} \frac{n}{4^n}(2x-1)^n$

15. $\sum_{n=1}^{\infty} (-1)^n \frac{(x-1)^n}{\sqrt{n}}$

16. $\sum_{n=1}^{\infty} \frac{(x-4)^n}{n 5^n}$

17. $\sum_{n=0}^{\infty} \frac{(-3)^n (x-1)^n}{\sqrt{n+1}}$

18. $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n^3}$

19. $\sum_{n=1}^{\infty} \frac{n x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

8.5 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $1, [-1, 1)$

2. $1, (-1, 1]$

3. $2, (-2, 2]$

4. $\frac{1}{5}, \left(-\frac{1}{5}, \frac{1}{5}\right)$

5. $1, (-1, 1)$

6. $1, [-1, 1]$

7. $\frac{1}{3}, \left[-\frac{1}{3}, \frac{1}{3}\right]$

8. $10, (-10, 10)$

9. $1, [-1, 1)$

10. $\infty, (-\infty, \infty)$

11. $\frac{1}{2}, \left[\frac{5}{2}, \frac{7}{2}\right)$

12. $1, [-2, 0]$

13. $\frac{1}{3}, \left(-1, -\frac{1}{3}\right)$

14. $2, \left(-\frac{3}{2}, \frac{5}{2}\right)$

15. $1, (0, 2]$

16. $5, [-1, 9)$

17. $\frac{1}{3}, \left(\frac{2}{3}, \frac{4}{3}\right]$

18. $\frac{1}{2}, [0, 1]$

19. $\infty, (-\infty, \infty)$

8.5 SOLUTIONS

E Click here for exercises.

"R" stands for "radius of convergence" and "I" stands for "interval of convergence" in this section.

1. If $a_n = \dfrac{x^n}{n+2}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+3}\cdot\frac{n+2}{x^n}\right| = |x|\lim_{n\to\infty}\frac{n+2}{n+3} = |x| < 1$$

for convergence (by the Ratio Test). So $R = 1$. When $x = 1$, the series is $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n+2}$ which diverges (Integral Test or Comparison Test), and when $x = -1$, it is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n+2}$ which converges (Alternating Series Test), so $I = [-1, 1)$.

2. If $a_n = \dfrac{(-1)^n x^n}{\sqrt[3]{n}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{1/3} = |x| < 1 \text{ for}$$

convergence (by the Ratio Test), and $R = 1$. When $x = 1$, $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{n}}$ which is a convergent alternating series, but when $x = -1$, $\displaystyle\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1}{n^{1/3}}$ which is a divergent p-series ($p = \frac{1}{3} < 1$), so $I = (-1, 1]$.

3. If $a_n = \dfrac{(-1)^n x^n}{n2^n}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}/\left[(n+1)\,2^{n+1}\right]}{x^n/(n2^n)}\right| = \left|\frac{x}{2}\right|\lim_{n\to\infty}\frac{n}{n+1} = \left|\frac{x}{2}\right| < 1$$

for convergence, so $|x| < 2$ and $R = 2$. When $x = 2$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ which converges by the Alternating Series Test. When $x = -2$, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ which diverges (harmonic series), so $I = (-2, 2]$.

4. If $a_n = n5^n x^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 5\,|x|\lim_{n\to\infty}\frac{n+1}{n} = 5\,|x| < 1 \text{ for}$$

convergence (by the Ratio Test), so $R = \frac{1}{5}$. If $x = \pm\frac{1}{5}$, $|a_n| = n \to \infty$ as $n \to \infty$, so $\sum_{n=1}^{\infty} a_n$ diverges by the Test for Divergence and $I = \left(-\frac{1}{5}, \frac{1}{5}\right)$.

5. If $a_n = nx^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\,x^{n+1}}{nx^n}\right| = |x|\lim_{n\to\infty}\frac{n+1}{n} = |x| < 1$$

for convergence (by the Ratio Test). So $R = 1$. When $x = 1$ or -1, $\lim_{n\to\infty} nx^n$ does not exist, so $\sum_{n=0}^{\infty} nx^n$ diverges for $x = \pm 1$. So $I = (-1, 1)$.

6. If $a_n = \dfrac{x^n}{n^2}$ then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = |x| < 1 \text{ for}$$

convergence (by the Ratio Test), so $R = 1$. If $x = \pm 1$, $\displaystyle\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{n^2}$, which converges ($p = 2 > 1$), so $I = [-1, 1]$.

7. If $a_n = \dfrac{3^n x^n}{(n+1)^2}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}x^{n+1}}{(n+2)^2}\cdot\frac{(n+1)^2}{3^n x^n}\right| = 3\,|x|\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^2 = 3\,|x| < 1$$

for convergence, so $|x| < \frac{1}{3}$ and $R = \frac{1}{3}$. When $x = \frac{1}{3}$, $\displaystyle\sum_{n=0}^{\infty}\frac{3^n x^n}{(n+1)^2} = \sum_{n=0}^{\infty}\frac{1}{(n+1)^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$, which is a convergent p-series ($p = 2 > 1$). When $x = -\frac{1}{3}$, $\displaystyle\sum_{n=0}^{\infty}\frac{3^n x^n}{(n+1)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}$, which converges by the Alternating Series Test, so $I = \left[-\frac{1}{3}, \frac{1}{3}\right]$.

8. If $a_n = \dfrac{n^2 x^n}{10^n}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{10}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = \frac{|x|}{10} < 1 \text{ for}$$

convergence (by the Ratio Test), so $R = 10$. If $x = \pm 10$, $|a_n| = n^2 \to \infty$ as $n \to \infty$, so $\sum_{n=0}^{\infty} a_n$ diverges (Test for Divergence) and $I = (-10, 10)$.

9. If $a_n = \dfrac{x^n}{\ln n}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{\ln(n+1)}\cdot\frac{\ln n}{x^n}\right| = |x|\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} \overset{\text{H}}{=} |x|$$

so $R = 1$. When $x = 1$, $\sum\limits_{n=2}^{\infty}\dfrac{x^n}{\ln n} = \sum\limits_{n=2}^{\infty}\dfrac{1}{\ln n}$, which diverges because $\dfrac{1}{\ln n} > \dfrac{1}{n}$ and $\sum\limits_{n=2}^{\infty}\dfrac{1}{n}$ is the divergent harmonic series. When $x = -1$, $\sum\limits_{n=2}^{\infty}\dfrac{x^n}{\ln n} = \sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{\ln n}$, which converges by the Alternating Series Test. So $I = [-1, 1)$.

10. If $a_n = \dfrac{(-1)^n x^{2n-1}}{(2n-1)!}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{x^2}{(2n+1)\,2n} = 0 < 1 \text{ for all } x.$$

By the Ratio Test the series converges for all x, so $R = \infty$ and $I = (-\infty, \infty)$.

11. If $a_n = \dfrac{2^n(x-3)^n}{n+3}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-3)^{n+1}}{n+4}\cdot\frac{n+3}{2^n(x-3)^n}\right| = 2|x-3|\lim_{n\to\infty}\frac{n+3}{n+4} = 2|x-3| < 1$$

for convergence, or $|x-3| < \frac{1}{2} \;\Leftrightarrow\; \frac{5}{2} < x < \frac{7}{2}$, and $R = \frac{1}{2}$. When $x = \frac{5}{2}$, $\sum\limits_{n=0}^{\infty}\dfrac{2^n(x-3)^n}{n+3} = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{n+3}$, which converges by the Alternating Series Test. When $x = \frac{7}{2}$, $\sum\limits_{n=0}^{\infty}\dfrac{2^n(x-3)^n}{n+3} = \sum\limits_{n=0}^{\infty}\dfrac{1}{n+3}$, similar to the harmonic series, which diverges. So $I = \left[\frac{5}{2}, \frac{7}{2}\right)$.

12. If $a_n = \dfrac{(x+1)^n}{n(n+1)}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x+1|\lim_{n\to\infty}\frac{n}{n+2} = |x+1| < 1$$

for convergence, or $-2 < x < 0$ and $R = 1$. If $x = -2$ or 0, then $|a_n| = \dfrac{1}{n^2+n} < \dfrac{1}{n^2}$, so $\sum\limits_{n=1}^{\infty}|a_n|$ converges since $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ does ($p = 2 > 1$), and $I = [-2, 0]$.

13. If $a_n = \sqrt{n}\,(3x+2)^n$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\sqrt{n+1}\,(3x+2)^{n+1}}{\sqrt{n}\,(3x+2)^n}\right| = \left|\sqrt{1+\frac{1}{n}}\cdot(3x+2)\right| \to |3x+2| \text{ as } n\to\infty$$

so for convergence, $|3x+2| < 1 \;\Rightarrow\; \left|x+\frac{2}{3}\right| < \frac{1}{3}$ so $R = \frac{1}{3}$ and $-1 < x < -\frac{1}{3}$. If $x = -1$, the series becomes $\sum\limits_{n=0}^{\infty}(-1)^n\sqrt{n}$ which is divergent by the Test for Divergence. If $x = -\frac{1}{3}$, the series is $\sum\limits_{n=0}^{\infty}\sqrt{n}$ which is also divergent by the Test for Divergence. So $I = \left(-1, -\frac{1}{3}\right)$.

14. If $a_n = \dfrac{n}{4^n}(2x-1)^n$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(2x-1)^{n+1}}{4^{n+1}}\cdot\frac{4^n}{n(2x-1)^n}\right| = \left|\frac{2x-1}{4}\left(1+\frac{1}{n}\right)\right| \to \tfrac{1}{2}\left|x-\tfrac{1}{2}\right| \text{ as } n\to\infty.$$

For convergence, $\frac{1}{2}\left|x-\frac{1}{2}\right| < 1 \;\Rightarrow\; \left|x-\frac{1}{2}\right| < 2 \;\Rightarrow\; R = 2$ and $-2 < x - \frac{1}{2} < 2 \;\Rightarrow\; -\frac{3}{2} < x < \frac{5}{2}$. If $x = -\frac{3}{2}$, the series becomes $\sum\limits_{n=0}^{\infty}\dfrac{n}{4^n}(-4)^n = \sum\limits_{n=0}^{\infty}(-1)^n n$ which is divergent by the Test for Divergence. If $x = \frac{5}{2}$, the series is $\sum\limits_{n=0}^{\infty}\dfrac{n}{4^n}4^n = \sum\limits_{n=0}^{\infty}n$, also divergent by the Test for Divergence. So $I = \left(-\frac{3}{2}, \frac{5}{2}\right)$.

15. If $a_n = \dfrac{(-1)^n(x-1)^n}{\sqrt{n}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(x-1)^n}\right| = |x-1|\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = |x-1| < 1$$

for convergence, or $0 < x < 2$, and $R = 1$. When $x = 0$, $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n(x-1)^n}{\sqrt{n}} = \sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$ which is a divergent p-series ($p = \frac{1}{2} < 1$). When $x = 2$, the series is $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{\sqrt{n}}$ which converges by the Alternating Series Test. So $I = (0, 2]$.

16. If $a_n = \dfrac{(x-4)^n}{n5^n}$ then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-4|}{5}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x-4|}{5} < 1$$

for convergence, or $-1 < x < 9$ and $R = 5$. When $x = 9$, $\sum\limits_{n=1}^{\infty}a_n = \sum\limits_{n=1}^{\infty}\dfrac{1}{n}$ which diverges (harmonic series), and when $x = -1$, $\sum\limits_{n=1}^{\infty}a_n = \sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n}$ which converges by the Alternating Series Test, so $I = [-1, 9)$.

17. If $a_n = \dfrac{(-3)^n (x-1)^n}{\sqrt{n+1}}$ then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 3\,|x-1|\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)^{1/2} = 3\,|x-1| < 1$$

for convergence, or $\frac{2}{3} < x < \frac{4}{3}$ and $R = \frac{1}{3}$. When $x = \frac{4}{3}$, $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ which is a convergent alternating series, and when $x = \frac{2}{3}$, $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n+1}}$ which is a divergent p-series ($p = \frac{1}{2} < 1$), so $I = \left(\frac{2}{3}, \frac{4}{3}\right]$.

18. If $a_n = \dfrac{(2x-1)^n}{n^3}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |2x-1|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^3 = |2x-1| < 1$$

for convergence, so $\left|x-\frac{1}{2}\right| < \frac{1}{2} \quad\Leftrightarrow\quad 0 < x < 1$, and $R = \frac{1}{2}$. The series $\sum_{n=1}^{\infty}\frac{(2x-1)^n}{n^3}$ converges both for $x = 0$ and $x = 1$ (in the first case because of the Alternating Series Test and in the second case because we get a p-series with $p = 3 > 1$). So $I = [0, 1]$.

19. If $a_n = \dfrac{nx^n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x|\lim_{n\to\infty}\frac{n+1}{n\,(2n+1)} = 0$$ for all x. So the series converges for all $x \quad\Rightarrow\quad R = \infty$ and $I = (-\infty, \infty)$.