

6.2 TRIGONOMETRIC INTEGRALS AND SUBSTITUTIONS

EXAMPLE A Evaluate $\int \frac{x}{\sqrt{3 - 2x - x^2}} dx$.

SOLUTION We can transform the integrand into a function for which trigonometric substitution is appropriate by first completing the square under the root sign:

$$\begin{aligned} 3 - 2x - x^2 &= 3 - (x^2 + 2x) = 3 + 1 - (x^2 + 2x + 1) \\ &= 4 - (x + 1)^2 \end{aligned}$$

This suggests that we make the substitution $u = x + 1$. Then $du = dx$ and $x = u - 1$, so

$$\int \frac{x}{\sqrt{3 - 2x - x^2}} dx = \int \frac{u - 1}{\sqrt{4 - u^2}} du$$

We now substitute $u = 2 \sin \theta$, giving $du = 2 \cos \theta d\theta$ and $\sqrt{4 - u^2} = 2 \cos \theta$, so

$$\begin{aligned} \int \frac{x}{\sqrt{3 - 2x - x^2}} dx &= \int \frac{2 \sin \theta - 1}{2 \cos \theta} 2 \cos \theta d\theta \\ &= \int (2 \sin \theta - 1) d\theta \\ &= -2 \cos \theta - \theta + C \\ &= -\sqrt{4 - u^2} - \sin^{-1}\left(\frac{u}{2}\right) + C \\ &= -\sqrt{3 - 2x - x^2} - \sin^{-1}\left(\frac{x + 1}{2}\right) + C \quad \blacksquare \end{aligned}$$