

6.3 PARTIAL FRACTIONS

EXAMPLE A Find $\int \frac{dx}{x^2 - a^2}$, where $a \neq 0$.

SOLUTION The method of partial fractions gives

$$\frac{1}{x^2 - a^2} = \frac{1}{(x - a)(x + a)} = \frac{A}{x - a} + \frac{B}{x + a}$$

and therefore

$$A(x + a) + B(x - a) = 1$$

Using the method of the note after Example 2, we put $x = a$ in this equation and get $A(2a) = 1$, so $A = 1/(2a)$. If we put $x = -a$, we get $B(-2a) = 1$, so $B = -1/(2a)$. Thus

$$\begin{aligned} \int \frac{dx}{x^2 - a^2} &= \frac{1}{2a} \int \left(\frac{1}{x - a} - \frac{1}{x + a} \right) dx \\ &= \frac{1}{2a} (\ln |x - a| - \ln |x + a|) + C \end{aligned}$$

Since $\ln x - \ln y = \ln(x/y)$, we can write the integral as

$$\mathbf{1} \quad \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C \quad \blacksquare$$

EXAMPLE B Evaluate $\int \frac{\sqrt{x+4}}{x} dx$.

SOLUTION Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$, so $x = u^2 - 4$ and $dx = 2u du$. Therefore

$$\begin{aligned} \int \frac{\sqrt{x+4}}{x} dx &= \int \frac{u}{u^2 - 4} 2u du = 2 \int \frac{u^2}{u^2 - 4} du \\ &= 2 \int \left(1 + \frac{4}{u^2 - 4} \right) du \end{aligned}$$

We can evaluate this integral either by factoring $u^2 - 4$ as $(u - 2)(u + 2)$ and using partial fractions or by using Formula 1 in Example A with $a = 2$:

$$\begin{aligned} \int \frac{\sqrt{x+4}}{x} dx &= 2 \int du + 8 \int \frac{du}{u^2 - 4} \\ &= 2u + 8 \cdot \frac{1}{2 \cdot 2} \ln \left| \frac{u - 2}{u + 2} \right| + C \\ &= 2\sqrt{x+4} + 2 \ln \left| \frac{\sqrt{x+4} - 2}{\sqrt{x+4} + 2} \right| + C \quad \blacksquare \end{aligned}$$