6.6 IMPROPER INTEGRALS

EXAMPLE A Evaluate $\int_0^1 \ln x \, dx$.

SOLUTION We know that the function $f(x) = \ln x$ has a vertical asymptote at 0 since $\lim_{x \to 0^+} \ln x = -\infty$. Thus, the given integral is improper and we have

$$\int_0^1 \ln x \, dx = \lim_{t \to 0^+} \int_t^1 \ln x \, dx$$

Now we integrate by parts with $u = \ln x$, $dv = dx$, $du = dx/x$, and $v = x$:

$$\begin{aligned} \int_t^1 \ln x \, dx &= x \ln x \Big]_t^1 - \int_t^1 dx \\ &= 1 \ln 1 - t \ln t - (1 - t) \\ &= -t \ln t - 1 + t \end{aligned}$$

To find the limit of the first term we use l'Hospital's Rule:

$$\begin{aligned} \lim_{t \to 0^+} t \ln t &= \lim_{t \to 0^+} \frac{\ln t}{1/t} = \lim_{t \to 0^+} \frac{1/t}{-1/t^2} \\ &= \lim_{t \to 0^+} (-t) = 0 \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^1 \ln x \, dx &= \lim_{t \to 0^+} (-t \ln t - 1 + t) \\ &= -0 - 1 + 0 = -1 \end{aligned}$$

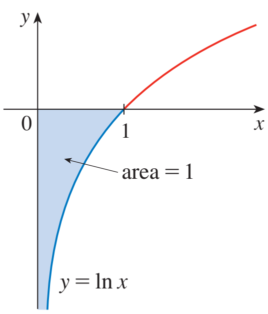

FIGURE 1

Figure 1 shows the geometric interpretation of this result. The area of the shaded region above $y = \ln x$ and below the x-axis is 1. ■