

7.7 DIFFERENTIAL EQUATIONS

EXAMPLE A Solve the initial-value problem $xy' = -y$, $x > 0$, $y(4) = 2$.

SOLUTION We write the differential equation as

$$x \frac{dy}{dx} = -y \quad \text{or} \quad \frac{dy}{y} = -\frac{dx}{x}$$

Therefore

$$\begin{aligned} \int \frac{dy}{y} &= -\int \frac{dx}{x} \\ \ln |y| &= -\ln |x| + C \\ |y| &= \frac{1}{|x|} e^C \\ y &= \frac{K}{x} \end{aligned}$$

where $K = \pm e^C$ is a constant. To determine K we put $x = 4$ and $y = 2$ in this equation:

$$2 = \frac{K}{4} \quad k = 8$$

The solution of the initial-value problem is

$$y = \frac{8}{x} \quad x > 0$$

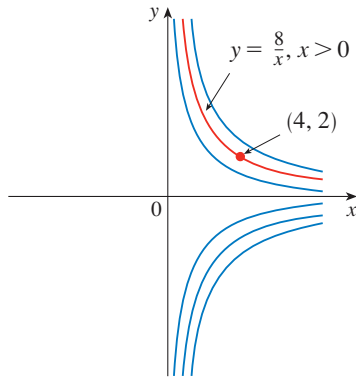


FIGURE 1

Figure 1 shows the family of solutions $xy = K$ for several values of K (equilateral hyperbolas) and, in particular, the solution that satisfies $y(4) = 2$ [the hyperbola that passes through the point $(4, 2)$]. ■

EXAMPLE B Solve $y' = 1 + y^2 - 2x - 2xy^2$, $y(0) = 0$, and graph the solution.

SOLUTION At first glance this does not look like a separable equation, but notice that it is possible to factor the right side as the product of a function of x and a function of y as follows:

$$\begin{aligned} \frac{dy}{dx} &= 1 + y^2 - 2x - 2xy^2 = (1 - 2x)(1 + y^2) \\ \int \frac{dy}{1 + y^2} &= \int (1 - 2x) dx \\ \tan^{-1} y &= x - x^2 + C \end{aligned}$$

Putting $x = 0$ and $y = 0$, we get $C = \tan^{-1} 0 = 0$, so

$$\tan^{-1} y = x - x^2$$

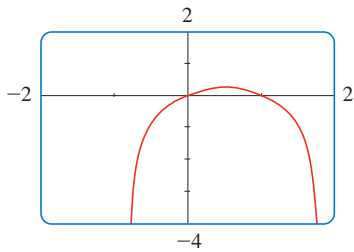


FIGURE 2

To graph this equation we notice that it is equivalent to

$$y = \tan(x - x^2)$$

provided that $-\pi/2 < x - x^2 < \pi/2$. Solving these inequalities using the quadratic formula, we find that

$$\frac{1}{2}(1 - \sqrt{1 + 2\pi}) < x < \frac{1}{2}(1 + \sqrt{1 + 2\pi})$$

This enables us to graph the solution in Figure 2. ■