

8.1 SEQUENCES

EXAMPLE A Is the sequence $a_n = \frac{n}{\sqrt{10+n}}$ convergent or divergent?

SOLUTION As in Example 4, we divide numerator and denominator by n :

$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt{10+n}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{\frac{10}{n^2} + \frac{1}{n}}} = \infty$$

because the numerator is constant and the denominator approaches 0. So $\{a_n\}$ is divergent. ■

EXAMPLE B Investigate the sequence $\{a_n\}$ defined by the *recurrence relation*

$$a_1 = 2 \quad a_{n+1} = \frac{1}{2}(a_n + 6) \quad \text{for } n = 1, 2, 3, \dots$$

SOLUTION We begin by computing the first several terms:

$$\begin{array}{lll} a_1 = 2 & a_2 = \frac{1}{2}(2 + 6) = 4 & a_3 = \frac{1}{2}(4 + 6) = 5 \\ a_4 = \frac{1}{2}(5 + 6) = 5.5 & a_5 = 5.75 & a_6 = 5.875 \\ a_7 = 5.9375 & a_8 = 5.96875 & a_9 = 5.984375 \end{array}$$

■ Mathematical induction is often used in dealing with recursive sequences. For a discussion of the Principle of Mathematical Induction see *Additional Topics: Principles of Problem Solving*.

These initial terms suggest that the sequence is increasing and the terms are approaching 6. To confirm that the sequence is increasing, we use mathematical induction to show that $a_{n+1} > a_n$ for all $n \geq 1$. This is true for $n = 1$ because $a_2 = 4 > a_1$. If we assume that it is true for $n = k$, then we have

$$a_{k+1} > a_k$$

so

$$a_{k+1} + 6 > a_k + 6$$

and

$$\frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6)$$

Thus

$$a_{k+2} > a_{k+1}$$

We have deduced that $a_{n+1} > a_n$ is true for $n = k + 1$. Therefore, the inequality is true for all n by induction.

Next we verify that $\{a_n\}$ is bounded by showing that $a_n < 6$ for all n . (Since the sequence is increasing, we already know that it has a lower bound: $a_n \geq a_1 = 2$ for all n .) We know that $a_1 < 6$, so the assertion is true for $n = 1$. Suppose it is true for $n = k$. Then

$$a_k < 6$$

so

$$a_k + 6 < 12$$

and

$$\frac{1}{2}(a_k + 6) < \frac{1}{2}(12) = 6$$

Thus

$$a_{k+1} < 6$$

This shows, by mathematical induction, that $a_n < 6$ for all n .

Since the sequence $\{a_n\}$ is increasing and bounded, the Monotonic Sequence Theorem guarantees that it has a limit. The theorem doesn't tell us what the value of the limit is. But now that we know $L = \lim_{n \rightarrow \infty} a_n$ exists, we can use the given recurrence relation to write

$$\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{2}(a_n + 6) = \frac{1}{2}(\lim_{n \rightarrow \infty} a_n + 6) = \frac{1}{2}(L + 6)$$

Since $a_n \rightarrow L$, it follows that $a_{n+1} \rightarrow L$ too (as $n \rightarrow \infty$, $n + 1 \rightarrow \infty$ also). So we have

$$L = \frac{1}{2}(L + 6)$$

Solving this equation for L , we get $L = 6$, as we predicted. ■