

## 8.7 TAYLOR AND MACLAURIN SERIES

**EXAMPLE A** Represent  $f(x) = \sin x$  as the sum of its Taylor series centered at  $\pi/3$ .

**SOLUTION** Arranging our work in columns, we have

$$\begin{array}{ll} f(x) = \sin x & f\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \\ f'(x) = \cos x & f'\left(\frac{\pi}{3}\right) = \frac{1}{2} \\ f''(x) = -\sin x & f''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \\ f'''(x) = -\cos x & f'''\left(\frac{\pi}{3}\right) = -\frac{1}{2} \end{array}$$

and this pattern repeats indefinitely. Therefore, the Taylor series at  $\pi/3$  is

$$\begin{aligned} f\left(\frac{\pi}{3}\right) + \frac{f'\left(\frac{\pi}{3}\right)}{1!} \left(x - \frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!} \left(x - \frac{\pi}{3}\right)^2 + \frac{f'''\left(\frac{\pi}{3}\right)}{3!} \left(x - \frac{\pi}{3}\right)^3 + \cdots \\ = \frac{\sqrt{3}}{2} + \frac{1}{2 \cdot 1!} \left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2 \cdot 2!} \left(x - \frac{\pi}{3}\right)^2 - \frac{1}{2 \cdot 3!} \left(x - \frac{\pi}{3}\right)^3 + \cdots \end{aligned}$$

The proof that this series represents  $\sin x$  for all  $x$  is very similar to that in Example 4. [Just replace  $x$  by  $x - \pi/3$  in (15).] We can write the series in sigma notation if we separate the terms that contain  $\sqrt{3}$ :

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{3}}{2(2n)!} \left(x - \frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2(2n+1)!} \left(x - \frac{\pi}{3}\right)^{2n+1}$$

■ We have obtained two different series representations for  $\sin x$ , the Maclaurin series in Example 4 and the Taylor series in Example A. It is best to use the Maclaurin series for values of  $x$  near 0 and the Taylor series for  $x$  near  $\pi/3$ . Notice that the third Taylor polynomial  $T_3$  in Figure 1 is a good approximation to  $\sin x$  near  $\pi/3$  but not as good near 0. Compare it with the third Maclaurin polynomial  $T_3$  in Figure 2, where the opposite is true.

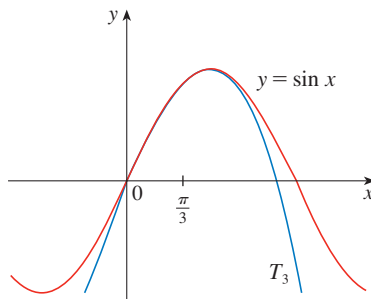


FIGURE 1

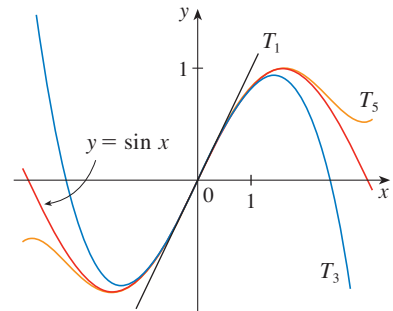


FIGURE 2

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▶ **EXAMPLE B** Expand  $\frac{1}{(1+x)^2}$  as a power series.

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**SOLUTION** We use the binomial series with  $k = -2$ . The binomial coefficient is

$$\begin{aligned} \binom{-2}{n} &= \frac{(-2)(-3)(-4) \cdots (-2-n+1)}{n!} \\ &= \frac{(-1)^n 2 \cdot 3 \cdot 4 \cdots n(n+1)}{n!} = (-1)^n (n+1) \end{aligned}$$

and so, when  $|x| < 1$ ,

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} x^n = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n = 1 - 2x + 3x^2 - 4x^3 + \cdots$$