9.2 CALCULUS WITH PARAMETRIC CURVES

EXAMPLE A Find an equation of the tangent line to the parametric curve

$$x = 2 \sin 2t \quad y = 2 \sin t$$

at the point $(\sqrt{3}, 1)$. Where does this curve have horizontal or vertical tangents?

SOLUTION At the point with parameter value t, the slope is

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{\dfrac{d}{dt}(2 \sin t)}{\dfrac{d}{dt}(2 \sin 2t)}$$

$$= \frac{2 \cos t}{2(\cos 2t)(2)} = \frac{\cos t}{2 \cos 2t}$$

The point $(\sqrt{3}, 1)$ corresponds to the parameter value $t = \pi/6$, so the slope of the tangent at that point is

$$\left.\frac{dy}{dx}\right|_{t=\pi/6} = \frac{\cos(\pi/6)}{2 \cos(\pi/3)} = \frac{\sqrt{3}/2}{2\left(\frac{1}{2}\right)} = \frac{\sqrt{3}}{2}$$

An equation of the tangent line is therefore

$$y - 1 = \frac{\sqrt{3}}{2}\left(x - \sqrt{3}\right) \qquad \text{or} \qquad y = \frac{\sqrt{3}}{2}x - \frac{1}{2}$$

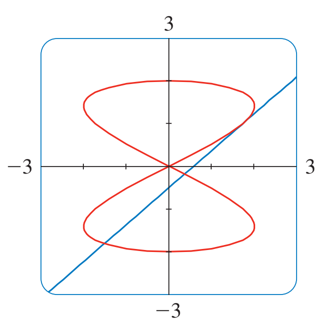

FIGURE 1

Figure 1 shows the curve and its tangent line.

The tangent line is horizontal when $dy/dx = 0$, which occurs when $\cos t = 0$ (and $\cos 2t \neq 0$), that is, when $t = \pi/2$ or $3\pi/2$. (Note that the entire curve is given by $0 \leqslant t \leqslant 2\pi$.) Thus, the curve has horizontal tangents at the points $(0, 2)$ and $(0, -2)$, which we could have guessed from Figure 1.

The tangent is vertical when $dx/dt = 4 \cos 2t = 0$ (and $\cos t \neq 0$), that is, when $t = \pi/4, 3\pi/4, 5\pi/4$, or $7\pi/4$. The corresponding four points on the curve are $\left(\pm 2, \pm\sqrt{2}\right)$. If we look again at Figure 1, we see that our answer appears to be reasonable. ■