12.7 TRIPLE INTEGRALS IN SPHERICAL COORDINATES

EXAMPLE A Find an equation in spherical coordinates for the hyperboloid of two sheets with equation $x^2 - y^2 - z^2 = 1$.

SOLUTION Substituting the expressions in Equations 3 into the given equation, we have

$$\rho^2 \sin^2\phi \cos^2\theta - \rho^2 \sin^2\phi \sin^2\theta - \rho^2 \cos^2\phi = 1$$

$$\rho^2[\sin^2\phi (\cos^2\theta - \sin^2\theta) - \cos^2\phi] = 1$$

or

$$\rho^2(\sin^2\phi \cos 2\theta - \cos^2\phi) = 1$$

■

EXAMPLE B Find a rectangular equation for the surface whose spherical equation is $\rho = \sin\theta \sin\phi$.

SOLUTION From Equations 2 and 1 we have

$$x^2 + y^2 + z^2 = \rho^2 = \rho \sin\theta \sin\phi = y$$

or

$$x^2 + \left(y - \tfrac{1}{2}\right)^2 + z^2 = \tfrac{1}{4}$$

which is the equation of a sphere with center $\left(0, \tfrac{1}{2}, 0\right)$ and radius $\tfrac{1}{2}$. ■

EXAMPLE C Use a computer to draw a picture of the solid that remains when a hole of radius 3 is drilled through the center of a sphere of radius 4.

▪ Most three-dimensional graphing programs can graph surfaces whose equa-tions are given in cylindrical or spherical coordinates. As Example C demonstrates, this is often the most convenient way of drawing a solid.

SOLUTION To keep the equations simple, let's choose the coordinate system so that the center of the sphere is at the origin and the axis of the cylinder that forms the hole is the z-axis. We could use either cylindrical or spherical coordinates to describe the solid, but the description is somewhat simpler if we use cylindrical coordinates. Then the equation of the cylinder is $r = 3$ and the equation of the sphere is $x^2 + y^2 + z^2 = 16$, or $r^2 + z^2 = 16$. The points in the solid lie outside the cylinder and inside the sphere, so they satisfy the inequalities

$$3 \leqslant r \leqslant \sqrt{16 - z^2}$$

To ensure that the computer graphs only the appropriate parts of these surfaces, we find where they intersect by solving the equations $r = 3$ and $r = \sqrt{16 - z^2}$:

$$\sqrt{16 - z^2} = 3 \quad \Rightarrow \quad 16 - z^2 = 9 \quad \Rightarrow \quad z^2 = 7 \quad \Rightarrow \quad z = \pm\sqrt{7}$$

The solid lies between $z = -\sqrt{7}$ and $z = \sqrt{7}$, so we ask the computer to graph the surfaces with the following equations and domains:

$$r = 3 \qquad 0 \leqslant \theta \leqslant 2\pi \qquad -\sqrt{7} \leqslant z \leqslant \sqrt{7}$$

$$r = \sqrt{16 - z^2} \qquad 0 \leqslant \theta \leqslant 2\pi \qquad -\sqrt{7} \leqslant z \leqslant \sqrt{7}$$

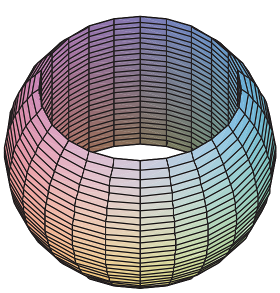

FIGURE 1

The resulting picture, shown in Figure 1, is exactly what we want. ■