CHALLENGE PROBLEMS

CHAPTER 3

A Click here for answers. S Click here for solutions.

1. Show that

$$\frac{d^n}{dx^n}(e^{ax}\sin bx) = r^n e^{ax}\sin(bx + n\theta)$$

where a and b are positive numbers, $r^2 = a^2 + b^2$, and $\theta = \tan^{-1}(b/a)$.

2. Show that $\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$.

3. If

$$y = \frac{x}{\sqrt{a^2 - 1}} - \frac{2}{\sqrt{a^2 - 1}} \arctan \frac{\sin x}{a + \sqrt{a^2 - 1} + \cos x}$$

show that $y' = \dfrac{1}{a + \cos x}$.

4. For which positive numbers a is it true that $a^x \geqslant 1 + x$ for all x?

5. For what value of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution?

6. A peach pie is taken out of the oven at 5:00 P.M. At that time it is piping hot: 100°C. At 5:10 P.M. its temperature is 80°C; at 5:20 P.M. it is 65°C. What is the temperature of the room?

ANSWERS

S Solutions

5. $2\sqrt{e}$

SOLUTIONS

E Exercises

1. Consider the statement that $\dfrac{d^n}{dx^n}\left(e^{ax}\sin bx\right) = r^n e^{ax}\sin(bx + n\theta)$. For $n = 1$,

$\dfrac{d}{dx}\left(e^{ax}\sin bx\right) = ae^{ax}\sin bx + be^{ax}\cos bx$, and

$$re^{ax}\sin(bx+\theta) = re^{ax}[\sin bx\cos\theta + \cos bx\sin\theta] = re^{ax}\left(\frac{a}{r}\sin bx + \frac{b}{r}\cos bx\right)$$
$$= ae^{ax}\sin bx + be^{ax}\cos bx$$

since $\tan\theta = \dfrac{b}{a} \quad\Rightarrow\quad \sin\theta = \dfrac{b}{r}$ and $\cos\theta = \dfrac{a}{r}$.

So the statement is true for $n = 1$. Assume it is true for $n = k$. Then

$$\frac{d^{k+1}}{dx^{k+1}}\left(e^{ax}\sin bx\right) = \frac{d}{dx}\left[r^k e^{ax}\sin(bx+k\theta)\right] = r^k ae^{ax}\sin(bx+k\theta) + r^k e^{ax} b\cos(bx+k\theta)$$
$$= r^k e^{ax}[a\sin(bx+k\theta) + b\cos(bx+k\theta)]$$

But

$$\sin[bx+(k+1)\theta] = \sin[(bx+k\theta)+\theta] = \sin(bx+k\theta)\cos\theta + \sin\theta\cos(bx+k\theta)$$
$$= \frac{a}{r}\sin(bx+k\theta) + \frac{b}{r}\cos(bx+k\theta)$$

Hence, $a\sin(bx+k\theta) + b\cos(bx+k\theta) = r\sin[bx+(k+1)\theta]$. So

$$\frac{d^{k+1}}{dx^{k+1}}\left(e^{ax}\sin bx\right) = r^k e^{ax}[a\sin(bx+k\theta) + b\cos(bx+k\theta)] = r^k e^{ax}[r\sin(bx+(k+1)\theta)]$$
$$= r^{k+1}e^{ax}[\sin(bx+(k+1)\theta)]$$

Therefore, the statement is true for all n by mathematical induction.

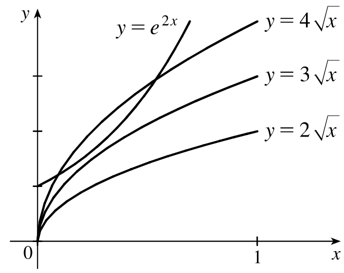
3. $y = \dfrac{x}{\sqrt{a^2-1}} - \dfrac{2}{\sqrt{a^2-1}}\arctan\dfrac{\sin x}{a+\sqrt{a^2-1}+\cos x}$. Let $k = a + \sqrt{a^2-1}$. Then

$$y' = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\cdot\frac{1}{1+\sin^2 x/(k+\cos x)^2}\cdot\frac{\cos x(k+\cos x)+\sin^2 x}{(k+\cos x)^2}$$
$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\cdot\frac{k\cos x+\cos^2 x+\sin^2 x}{(k+\cos x)^2+\sin^2 x} = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\cdot\frac{k\cos x+1}{k^2+2k\cos x+1}$$
$$= \frac{k^2+2k\cos x+1-2k\cos x-2}{\sqrt{a^2-1}\,(k^2+2k\cos x+1)} = \frac{k^2-1}{\sqrt{a^2-1}\,(k^2+2k\cos x+1)}$$

But $k^2 = 2a^2 + 2a\sqrt{a^2-1} - 1 = 2a\left(a+\sqrt{a^2-1}\right) - 1 = 2ak - 1$, so $k^2 + 1 = 2ak$, and

$k^2 - 1 = 2(ak-1)$. So $y' = \dfrac{2(ak-1)}{\sqrt{a^2-1}\,(2ak+2k\cos x)} = \dfrac{ak-1}{\sqrt{a^2-1}k\,(a+\cos x)}$. But

$ak - 1 = a^2 + a\sqrt{a^2-1} - 1 = k\sqrt{a^2-1}$, so $y' = 1/(a+\cos x)$.



5. Let $f(x) = e^{2x}$ and $g(x) = k\sqrt{x}$ $(k > 0)$. From the graphs of f and g, we see that f will intersect g exactly once when f and g share a tangent line. Thus, we must have $f = g$ and $f' = g'$ at $x = a$.

$f(a) = g(a) \quad \Rightarrow \quad e^{2a} = k\sqrt{a}$ **(1)** and $f'(a) = g'(a) \quad \Rightarrow$

$2e^{2a} = \dfrac{k}{2\sqrt{a}} \quad \Rightarrow \quad e^{2a} = \dfrac{k}{4\sqrt{a}}$. So we must have $k\sqrt{a} = \dfrac{k}{4\sqrt{a}}$

$\Rightarrow \quad (\sqrt{a})^2 = \dfrac{k}{4k} \quad \Rightarrow \quad a = \frac{1}{4}$. From **(1)**, $e^{2(1/4)} = k\sqrt{1/4} \quad \Rightarrow \quad k = 2e^{1/2} = 2\sqrt{e} \approx 3.297$.