CHALLENGE PROBLEMS

CHAPTER 9

A Click here for answers. S Click here for solutions.

1. A curve is defined by the parametric equations

$$x = \int_1^t \frac{\cos u}{u}\, du \qquad y = \int_1^t \frac{\sin u}{u}\, du$$

Find the length of the arc of the curve from the origin to the nearest point where there is a vertical tangent line.

2. (a) Find the highest and lowest points on the curve $x^4 + y^4 = x^2 + y^2$.
(b) Sketch the curve. (Notice that it is symmetric with respect to both axes and both of the lines $y = \pm x$, so it suffices to consider $y \geqslant x \geqslant 0$ initially.)
CAS (c) Use polar coordinates and a computer algebra system to find the area enclosed by the curve.

3. What is the smallest viewing rectangle that contains every member of the family of polar curves $r = 1 + c \sin\theta$, where $0 \leqslant c \leqslant 1$? Illustrate your answer by graphing several members of the family in this viewing rectangle.

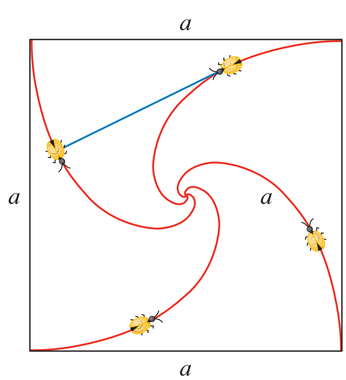


FIGURE FOR PROBLEM 4

4. Four bugs are placed at the four corners of a square with side length a. The bugs crawl counterclockwise at the same speed and each bug crawls directly toward the next bug at all times. They approach the center of the square along spiral paths.
(a) Find the polar equation of a bug's path assuming the pole is at the center of the square. (Use the fact that the line joining one bug to the next is tangent to the bug's path.)
(b) Find the distance traveled by a bug by the time it meets the other bugs at the center.

5. A curve called the **folium of Descartes** is defined by the parametric equations

$$x = \frac{3t}{1 + t^3} \qquad y = \frac{3t^2}{1 + t^3}$$

(a) Show that if (a, b) lies on the curve, then so does (b, a); that is, the curve is symmetric with respect to the line $y = x$. Where does the curve intersect this line?
(b) Find the points on the curve where the tangent lines are horizontal or vertical.
(c) Show that the line $y = -x - 1$ is a slant asymptote.
(d) Sketch the curve.
(e) Show that a Cartesian equation of this curve is $x^3 + y^3 = 3xy$.
(f) Show that the polar equation can be written in the form

$$r = \frac{3 \sec\theta \tan\theta}{1 + \tan^3\theta}$$

(g) Find the area enclosed by the loop of this curve.
CAS (h) Show that the area of the loop is the same as the area that lies between the asymptote and the infinite branches of the curve. (Use a computer algebra system to evaluate the integral.)

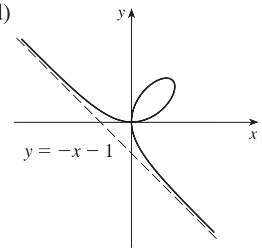
ANSWERS

S Solutions

1. $\ln(\pi/2)$ **3.** $\left[-\frac{3}{4}\sqrt{3}, \frac{3}{4}\sqrt{3}\right] \times [-1, 2]$

5. (a) At (0, 0) and $\left(\frac{3}{2}, \frac{3}{2}\right)$ (b) Horizontal tangents at (0, 0) and $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$; vertical tangents at (0, 0) and $\left(\sqrt[3]{4}, \sqrt[3]{2}\right)$

(d)



(g) $\frac{3}{2}$

SOLUTIONS

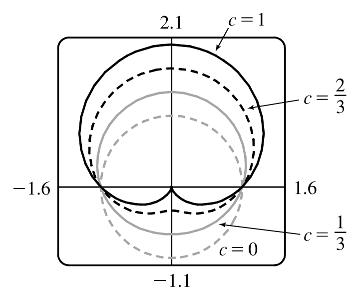
E Exercises

1. $x = \int_1^t \frac{\cos u}{u}\,du$, $y = \int_1^t \frac{\sin u}{u}\,du$, so by FTC1, we have $\frac{dx}{dt} = \frac{\cos t}{t}$ and $\frac{dy}{dt} = \frac{\sin t}{t}$. Vertical tangent lines occur when $\frac{dx}{dt} = 0 \quad \Leftrightarrow \quad \cos t = 0$. The parameter value corresponding to $(x, y) = (0, 0)$ is $t = 1$, so the nearest vertical tangent occurs when $t = \frac{\pi}{2}$. Therefore, the arc length between these points is

$$L = \int_1^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_1^{\pi/2} \sqrt{\frac{\cos^2 t}{t^2} + \frac{\sin^2 t}{t^2}}\,dt = \int_1^{\pi/2} \frac{dt}{t} = \Big[\ln t\Big]_1^{\pi/2} = \ln \frac{\pi}{2}$$

3. In terms of x and y, we have $x = r\cos\theta = (1 + c\sin\theta)\cos\theta = \cos\theta + c\sin\theta\cos\theta = \cos\theta + \frac{1}{2}c\sin 2\theta$ and $y = r\sin\theta = (1 + c\sin\theta)\sin\theta = \sin\theta + c\sin^2\theta$. Now $-1 \le \sin\theta \le 1 \quad \Rightarrow$ $-1 \le \sin\theta + c\sin^2\theta \le 1 + c \le 2$, so $-1 \le y \le 2$. Furthermore, $y = 2$ when $c = 1$ and $\theta = \frac{\pi}{2}$, while $y = -1$ for $c = 0$ and $\theta = \frac{3\pi}{2}$. Therefore, we need a viewing rectangle with $-1 \le y \le 2$.

To find the x-values, look at the equation $x = \cos\theta + \frac{1}{2}c\sin 2\theta$ and use the fact that $\sin 2\theta \ge 0$ for $0 \le \theta \le \frac{\pi}{2}$ and $\sin 2\theta \le 0$ for $-\frac{\pi}{2} \le \theta \le 0$. [Because $r = 1 + c\sin\theta$ is symmetric about the y-axis, we only need to consider $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.] So for $-\frac{\pi}{2} \le \theta \le 0$, x has a maximum value when $c = 0$ and then $x = \cos\theta$ has a maximum value of 1 at $\theta = 0$. Thus, the maximum value of x must occur on $\left[0, \frac{\pi}{2}\right]$ with $c = 1$. Then $x = \cos\theta + \frac{1}{2}\sin 2\theta \quad \Rightarrow$ $\frac{dx}{d\theta} = -\sin\theta + \cos 2\theta = -\sin\theta + 1 - 2\sin^2\theta \quad \Rightarrow \quad \frac{dx}{d\theta} = -(2\sin\theta - 1)(\sin\theta + 1) = 0$ when $\sin\theta = -1$ or $\frac{1}{2}$ (but $\sin\theta \ne -1$ for $0 \le \theta \le \frac{\pi}{2}$). If $\sin\theta = \frac{1}{2}$, then $\theta = \frac{\pi}{6}$ and $x = \cos\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3} = \frac{3}{4}\sqrt{3}$. Thus, the maximum value of x is $\frac{3}{4}\sqrt{3}$, and, by symmetry, the minimum value is $-\frac{3}{4}\sqrt{3}$. Therefore, the smallest viewing rectangle that contains every member of the family of polar curves $r = 1 + c\sin\theta$, where $0 \le c \le 1$, is $\left[-\frac{3}{4}\sqrt{3}, \frac{3}{4}\sqrt{3}\right] \times [-1, 2]$.



5. (a) If (a, b) lies on the curve, then there is some parameter value t_1 such that $\frac{3t_1}{1 + t_1^3} = a$ and $\frac{3t_1^2}{1 + t_1^3} = b$. If $t_1 = 0$, the point is $(0, 0)$, which lies on the line $y = x$. If $t_1 \ne 0$, then the point corresponding to $t = \frac{1}{t_1}$ is given by $x = \frac{3(1/t_1)}{1 + (1/t_1)^3} = \frac{3t_1^2}{t_1^3 + 1} = b$, $y = \frac{3(1/t_1)^2}{1 + (1/t_1)^3} = \frac{3t_1}{t_1^3 + 1} = a$. So (b, a) also lies on the curve. [Another way to see this is to do part (e) first; the result is immediate.] The curve intersects the line $y = x$ when $\frac{3t}{1 + t^3} = \frac{3t^2}{1 + t^3} \quad \Rightarrow \quad t = t^2 \quad \Rightarrow \quad t = 0$ or 1, so the points are $(0, 0)$ and $\left(\frac{3}{2}, \frac{3}{2}\right)$.

(b) $\frac{dy}{dt} = \frac{(1 + t^3)(6t) - 3t^2(3t^2)}{(1 + t^3)^2} = \frac{6t - 3t^4}{(1 + t^3)^2} = 0$ when $6t - 3t^4 = 3t(2 - t^3) = 0 \quad \Rightarrow \quad t = 0$ or $t = \sqrt[3]{2}$, so there are horizontal tangents at $(0, 0)$ and $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using the symmetry from part (a), we see that there are vertical tangents at $(0, 0)$ and $\left(\sqrt[3]{4}, \sqrt[3]{2}\right)$.

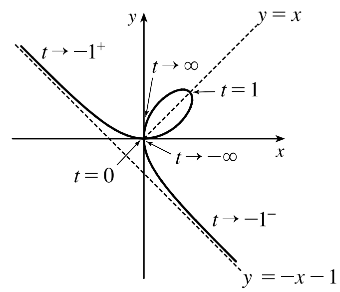
(c) Notice that as $t \to -1^+$, we have $x \to -\infty$ and $y \to \infty$. As $t \to -1^-$, we have $x \to \infty$ and $y \to -\infty$.

Also $y - (-x - 1) = y + x + 1 = \dfrac{3t + 3t^2 + (1 + t^3)}{1 + t^3} = \dfrac{(t+1)^3}{1 + t^3} = \dfrac{(t+1)^2}{t^2 - t + 1} \to 0$ as $t \to -1$.

So $y = -x - 1$ is a slant asymptote.

(d) $\dfrac{dx}{dt} = \dfrac{(1 + t^3)(3) - 3t(3t^2)}{(1 + t^3)^2} = \dfrac{3 - 6t^3}{(1 + t^3)^2}$ and from part (b) we have $\dfrac{dy}{dt} = \dfrac{6t - 3t^4}{(1 + t^3)^2}$. So $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{t(2 - t^3)}{1 - 2t^3}$. Also

$\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt} = \dfrac{2(1 + t^3)^4}{3(1 - 2t^3)^3} > 0 \quad \Leftrightarrow \quad t < \frac{1}{\sqrt[3]{2}}$. So the curve is concave upward there and has a minimum point at $(0, 0)$ and a maximum point at $\left(\sqrt[3]{2}, \sqrt[3]{4}\right)$. Using this together with the information from parts (a), (b), and (c), we sketch the curve.



(e) $x^3 + y^3 = \left(\dfrac{3t}{1 + t^3}\right)^3 + \left(\dfrac{3t^2}{1 + t^3}\right)^3 = \dfrac{27t^3 + 27t^6}{(1 + t^3)^3} = \dfrac{27t^3(1 + t^3)}{(1 + t^3)^3} = \dfrac{27t^3}{(1 + t^3)^2}$ and

$3xy = 3\left(\dfrac{3t}{1 + t^3}\right)\left(\dfrac{3t^2}{1 + t^3}\right) = \dfrac{27t^3}{(1 + t^3)^2}$, so $x^3 + y^3 = 3xy$.

(f) We start with the equation from part (e) and substitute $x = r\cos\theta$, $y = r\sin\theta$. Then $x^3 + y^3 = 3xy \quad \Rightarrow$ $r^3\cos^3\theta + r^3\sin^3\theta = 3r^2\cos\theta\sin\theta$. For $r \neq 0$, this gives $r = \dfrac{3\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}$. Dividing numerator and denominator by $\cos^3\theta$, we obtain $r = \dfrac{3\left(\dfrac{1}{\cos\theta}\right)\dfrac{\sin\theta}{\cos\theta}}{1 + \dfrac{\sin^3\theta}{\cos^3\theta}} = \dfrac{3\sec\theta\tan\theta}{1 + \tan^3\theta}$.

(g) The loop corresponds to $\theta \in \left(0, \frac{\pi}{2}\right)$, so its area is

$$A = \int_0^{\pi/2} \frac{r^2}{2}\,d\theta = \frac{1}{2}\int_0^{\pi/2}\left(\frac{3\sec\theta\tan\theta}{1 + \tan^3\theta}\right)^2 d\theta = \frac{9}{2}\int_0^{\pi/2}\frac{\sec^2\theta\tan^2\theta}{(1 + \tan^3\theta)^2}\,d\theta$$

$$= \frac{9}{2}\int_0^{\infty}\frac{u^2\,du}{(1 + u^3)^2} \quad [\text{let } u = \tan\theta] \quad = \lim_{b\to\infty}\tfrac{9}{2}\left[-\tfrac{1}{3}\left(1 + u^3\right)^{-1}\right]_0^b = \tfrac{3}{2}$$

(h) By symmetry, the area between the folium and the line $y = -x - 1$ is equal to the enclosed area in the third quadrant, plus twice the enclosed area in the fourth quadrant. The area in the third quadrant is $\frac{1}{2}$, and since $y = -x - 1 \quad \Rightarrow \quad r\sin\theta = -r\cos\theta - 1 \quad \Rightarrow \quad r = -\dfrac{1}{\sin\theta + \cos\theta}$, the area in the fourth quadrant is

$$\frac{1}{2}\int_{-\pi/2}^{-\pi/4}\left[\left(-\frac{1}{\sin\theta + \cos\theta}\right)^2 - \left(\frac{3\sec\theta\tan\theta}{1 + \tan^3\theta}\right)^2\right]d\theta \overset{\text{CAS}}{=} \frac{1}{2}.$$ Therefore, the total area is $\frac{1}{2} + 2\left(\frac{1}{2}\right) = \frac{3}{2}$.