# CHALLENGE PROBLEMS

## CHAPTER 12

A Click here for answers. S Click here for solutions.

**1.** If $[\![x]\!]$ denotes the greatest integer in $x$, evaluate the integral

$$\iint_R [\![x + y]\!]\, dA$$

where $R = \{(x, y) \mid 1 \leqslant x \leqslant 3,\ 2 \leqslant y \leqslant 5\}$.

**2.** Evaluate the integral

$$\int_0^1 \int_0^1 e^{\max\{x^2, y^2\}}\, dy\, dx$$

where $\max\{x^2, y^2\}$ means the larger of the numbers $x^2$ and $y^2$.

**3.** Find the average value of the function $f(x) = \int_x^1 \cos(t^2)\, dt$ on the interval $[0, 1]$.

**4.** If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are constant vectors, $\mathbf{r}$ is the position vector $x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and $E$ is given by the inequalities $0 \leqslant \mathbf{a} \cdot \mathbf{r} \leqslant \alpha,\ 0 \leqslant \mathbf{b} \cdot \mathbf{r} \leqslant \beta,\ 0 \leqslant \mathbf{c} \cdot \mathbf{r} \leqslant \gamma$, show that

$$\iiint_E (\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})(\mathbf{c} \cdot \mathbf{r})\, dV = \frac{(\alpha\beta\gamma)^2}{8|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}$$

**5.** The double integral $\int_0^1 \int_0^1 \frac{1}{1 - xy}\, dx\, dy$ is an improper integral and could be defined as the limit of double integrals over the rectangle $[0, t] \times [0, t]$ as $t \to 1^-$. But if we expand the integrand as a geometric series, we can express the integral as the sum of an infinite series. Show that

$$\int_0^1 \int_0^1 \frac{1}{1 - xy}\, dx\, dy = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

**6.** Leonhard Euler was able to find the exact sum of the series in Problem 5. In 1736 he proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

In this problem we ask you to prove this fact by evaluating the double integral in Problem 5. Start by making the change of variables

$$x = \frac{u - v}{\sqrt{2}} \qquad y = \frac{u + v}{\sqrt{2}}$$

This gives a rotation about the origin through the angle $\pi/4$. You will need to sketch the corresponding region in the $uv$-plane.

[*Hint:* If, in evaluating the integral, you encounter either of the expressions $(1 - \sin\theta)/\cos\theta$ or $(\cos\theta)/(1 + \sin\theta)$, you might like to use the identity $\cos\theta = \sin((\pi/2) - \theta)$ and the corresponding identity for $\sin\theta$.]

**7.** (a) Show that

$$\int_0^1 \int_0^1 \int_0^1 \frac{1}{1 - xyz}\, dx\, dy\, dz = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

(Nobody has ever been able to find the exact value of the sum of this series.)

(b) Show that

$$\int_0^1 \int_0^1 \int_0^1 \frac{1}{1 + xyz}\, dx\, dy\, dz = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$$

Use this equation to evaluate the triple integral correct to two decimal places.

**8.** Show that

$$\int_0^\infty \frac{\arctan \pi x - \arctan x}{x}\, dx = \frac{\pi}{2} \ln \pi$$

by first expressing the integral as an iterated integral.

**9.** If $f$ is continuous, show that

$$\int_0^x \int_0^y \int_0^z f(t)\, dt\, dz\, dy = \tfrac{1}{2} \int_0^x (x - t)^2 f(t)\, dt$$

**10.** (a) A lamina has constant density $\rho$ and takes the shape of a disk with center the origin and radius $R$. Use Newton's Law of Gravitation (see Section 10.9) to show that the magnitude of the force of attraction that the lamina exerts on a body with mass $m$ located at the point $(0, 0, d)$ on the positive $z$-axis is

$$F = 2\pi G m \rho d \left( \frac{1}{d} - \frac{1}{\sqrt{R^2 + d^2}} \right)$$

[*Hint:* Divide the disk as in Figure 4 in Section 12.3 and first compute the vertical component of the force exerted by the polar subrectangle $R_{ij}$.]

(b) Show that the magnitude of the force of attraction of a lamina with density $\rho$ that occupies an entire plane on an object with mass $m$ located at a distance $d$ from the plane is

$$F = 2\pi G m \rho$$

Notice that this expression does not depend on $d$.

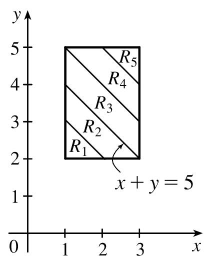
## ANSWERS

**S Solutions** **1.** 30 **3.** $\frac{1}{2}\sin 1$ **7.** (b) 0.90

## SOLUTIONS

**E Exercises**

**1.**




Let $R = \bigcup_{i=1}^{5} R_i$, where

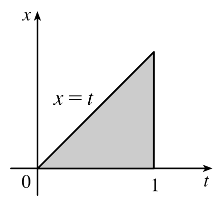
$R_i = \{(x,y) \mid x + y \geq i + 2, x + y < i + 3, 1 \leq x \leq 3, 2 \leq y \leq 5\}$.

$\iint_R [\![x+y]\!]\,dA = \sum_{i=1}^{5} \iint_{R_i} [\![x+y]\!]\,dA = \sum_{i=1}^{5} [\![x+y]\!] \iint_{R_i} dA$, since

$[\![x+y]\!] = \text{constant} = i + 2$ for $(x,y) \in R_i$. Therefore

$$\begin{aligned}\iint_R [\![x+y]\!]\,dA &= \textstyle\sum_{i=1}^{5} (i+2)\,[A(R_i)] \\ &= 3A(R_1) + 4A(R_2) + 5A(R_3) + 6A(R_4) + 7A(R_5) \\ &= 3\left(\tfrac{1}{2}\right) + 4\left(\tfrac{3}{2}\right) + 5(2) + 6\left(\tfrac{3}{2}\right) + 7\left(\tfrac{1}{2}\right) = 30\end{aligned}$$

**3.** $$\begin{aligned} f_{\text{ave}} &= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{1-0}\int_0^1 \left[\int_x^1 \cos(t^2)\,dt\right] dx \\ &= \textstyle\int_0^1 \int_x^1 \cos(t^2)\,dt\,dx \\ &= \textstyle\int_0^1 \int_0^t \cos(t^2)\,dx\,dt \qquad \text{[changing the order of integration]} \\ &= \textstyle\int_0^1 t\cos(t^2)\,dt = \tfrac{1}{2}\sin\left(t^2\right)\Big]_0^1 = \tfrac{1}{2}\sin 1 \end{aligned}$$




**5.** Since $|xy| < 1$, except at $(1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1-xy} = \sum\limits_{n=0}^{\infty} (xy)^n$, so

$$\begin{aligned} \int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy &= \int_0^1\int_0^1 \sum_{n=0}^{\infty}(xy)^n\,dx\,dy = \sum_{n=0}^{\infty}\int_0^1\int_0^1 (xy)^n\,dx\,dy \\ &= \sum_{n=0}^{\infty}\left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right] = \sum_{n=0}^{\infty}\frac{1}{n+1}\cdot\frac{1}{n+1} \\ &= \sum_{n=0}^{\infty}\frac{1}{(n+1)^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=0}^{\infty}\frac{1}{n^2} \end{aligned}$$

**7.** (a) Since $|xyz| < 1$ except at $(1,1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1-xyz} = \sum\limits_{n=0}^{\infty}(xyz)^n$, so

$$\begin{aligned} \int_0^1\int_0^1\int_0^1 \frac{1}{1-xyz}\,dx\,dy\,dz &= \int_0^1\int_0^1\int_0^1 \sum_{n=0}^{\infty}(xyz)^n\,dx\,dy\,dz = \sum_{n=0}^{\infty}\int_0^1\int_0^1\int_0^1 (xyz)^n\,dx\,dy\,dz \\ &= \sum_{n=0}^{\infty}\left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]\left[\int_0^1 z^n\,dz\right] \\ &= \sum_{n=0}^{\infty}\frac{1}{n+1}\cdot\frac{1}{n+1}\cdot\frac{1}{n+1} \\ &= \sum_{n=0}^{\infty}\frac{1}{(n+1)^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = \sum_{n=1}^{\infty}\frac{1}{n^3} \end{aligned}$$

(b) Since $|-xyz| < 1$, except at $(1, 1, 1)$, the formula for the sum of a geometric series gives

$\dfrac{1}{1+xyz} = \displaystyle\sum_{n=0}^{\infty}(-xyz)^n$, so

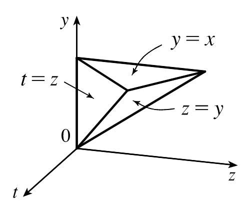
$$\begin{aligned}
\int_0^1\int_0^1\int_0^1 \frac{1}{1+xyz}\,dx\,dy\,dz &= \int_0^1\int_0^1\int_0^1 \sum_{n=0}^{\infty}(-xyz)^n\,dx\,dy\,dz \\
&= \sum_{n=0}^{\infty}\int_0^1\int_0^1\int_0^1 (-xyz)^n\,dx\,dy\,dz \\
&= \sum_{n=0}^{\infty}(-1)^n\left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]\left[\int_0^1 z^n\,dz\right] \\
&= \sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}\cdot\frac{1}{n+1}\cdot\frac{1}{n+1} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3} = \frac{1}{1^3}-\frac{1}{2^3}+\frac{1}{3^3}-\cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}
\end{aligned}$$

To evaluate this sum, we first write out a few terms: $s = 1 - \dfrac{1}{2^3} + \dfrac{1}{3^3} - \dfrac{1}{4^3} + \dfrac{1}{5^3} - \dfrac{1}{6^3} \approx 0.8998$.

Notice that $a_7 = \dfrac{1}{7^3} < 0.003$. By the Alternating Series Estimation Theorem from Section 8.4, we have $|s - s_6| \le a_7 < 0.003$. This error of 0.003 will not affect the second decimal place, so we have $s \approx 0.90$.

**9.** $\int_0^x\int_0^y\int_0^z f(t)\,dt\,dz\,dy = \iiint_E f(t)\,dV$, where $E = \{(t, z, y) \mid 0 \le t \le z, 0 \le z \le y, 0 \le y \le x\}$.

If we let $D$ be the projection of $E$ on the $yt$-plane then $D = \{(y, t) \mid 0 \le t \le x, t \le y \le x\}$. And we see from the diagram that $E = \{(t, z, y) \mid t \le z \le y, t \le y \le x, 0 \le t \le x\}$. So




$$\begin{aligned}
\int_0^x\int_0^y\int_0^z f(t)\,dt\,dz\,dy &= \int_0^x\int_t^x\int_t^y f(t)\,dz\,dy\,dt = \int_0^x\left[\int_t^x (y-t)\,f(t)\,dy\right]dt \\
&= \int_0^x\left[\left(\tfrac{1}{2}y^2 - ty\right)f(t)\right]_{y=t}^{y=x}dt = \int_0^x\left[\tfrac{1}{2}x^2 - tx - \tfrac{1}{2}t^2 + t^2\right]f(t)\,dt \\
&= \int_0^x\left[\tfrac{1}{2}x^2 - tx + \tfrac{1}{2}t^2\right]f(t)\,dt = \int_0^x\left(\tfrac{1}{2}x^2 - 2tx + t^2\right)f(t)\,dt \\
&= \tfrac{1}{2}\int_0^x (x-t)^2\,f(t)\,dt
\end{aligned}$$