4.3 DERIVATIVES AND THE SHAPES OF GRAPHS

EXAMPLE A Figure 1 shows a population graph for Cyprian honeybees raised in an apiary. How does the rate of population increase change over time? When is this rate highest? Over what intervals is P concave upward or concave downward?

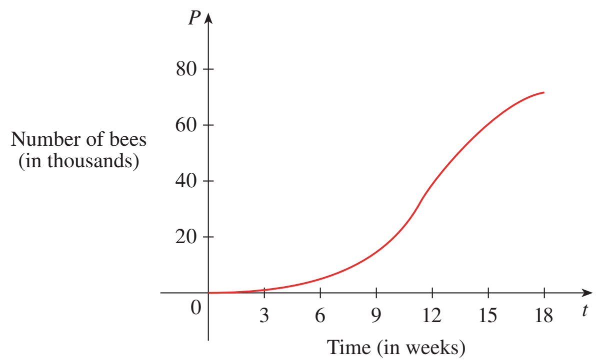


FIGURE 1

SOLUTION By looking at the slope of the curve as t increases, we see that the rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about $t = 12$ weeks, and decreases as the population begins to level off. As the population approaches its maximum value of about 75,000 (called the *carrying capacity*), the rate of increase, $P'(t)$, approaches 0. The curve appears to be concave upward on (0, 12) and concave downward on (12, 18). ■

EXAMPLE B Use the first and second derivatives of $f(x) = e^{1/x}$, together with asymptotes, to sketch its graph.

SOLUTION Notice that the domain of f is $\{x \mid x \neq 0\}$, so we check for vertical asymptotes by computing the left and right limits as $x \to 0$. As $x \to 0^+$, we know that $t = 1/x \to \infty$, so

$$\lim_{x \to 0^+} e^{1/x} = \lim_{t \to \infty} e^t = \infty$$

and this shows that $x = 0$ is a vertical asymptote. As $x \to 0^-$, we have $t = 1/x \to -\infty$, so

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$

TEC In Module 4.3 you can practice using graphical information about f' to determine the shape of the graph of f.

As $x \to \pm\infty$, we have $1/x \to 0$ and so

$$\lim_{x \to \pm\infty} e^{1/x} = e^0 = 1$$

This shows that $y = 1$ is a horizontal asymptote.

Now let's compute the derivative. The Chain Rule gives

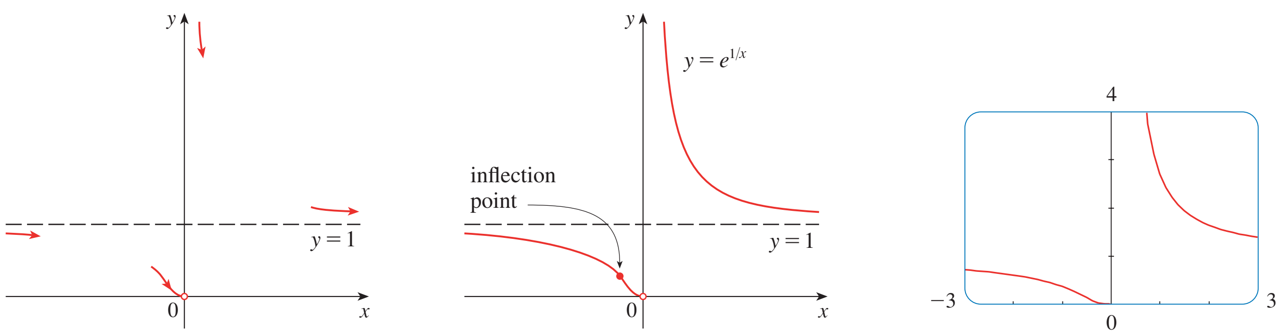
$$f'(x) = -\frac{e^{1/x}}{x^2}$$

Since $e^{1/x} > 0$ and $x^2 > 0$ for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$. Thus, f is decreasing on $(-\infty, 0)$ and on $(0, \infty)$. There is no critical number, so the function has no maximum or minimum. The second derivative is

$$f''(x) = -\frac{x^2 e^{1/x}(-1/x^2) - e^{1/x}(2x)}{x^4} = \frac{e^{1/x}(2x + 1)}{x^4}$$

Since $e^{1/x} > 0$ and $x^4 > 0$, we have $f''(x) > 0$ when $x > -\frac{1}{2}$ $(x \neq 0)$ and $f''(x) < 0$ when $x < -\frac{1}{2}$. So the curve is concave downward on $\left(-\infty, -\frac{1}{2}\right)$ and concave upward on $\left(-\frac{1}{2}, 0\right)$ and on $(0, \infty)$. The inflection point is $\left(-\frac{1}{2}, e^{-2}\right)$.

To sketch the graph of f we first draw the horizontal asymptote $y = 1$ (as a dashed line), together with the parts of the curve near the asymptotes in a preliminary sketch [Figure 2(a)]. These parts reflect the information concerning limits and the fact that f is decreasing on both $(-\infty, 0)$ and $(0, \infty)$. Notice that we have indicated that $f(x) \to 0$ as $x \to 0^-$ even though $f(0)$ does not exist. In Figure 2(b) we finish the sketch by incorporating the information concerning concavity and the inflection point. In Figure 2(c) we check our work with a graphing device.



(a) Preliminary sketch (b) Finished sketch (c) Computer confirmation

FIGURE 2 ■

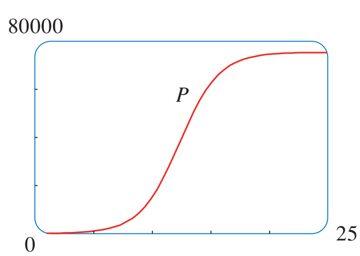


FIGURE 3

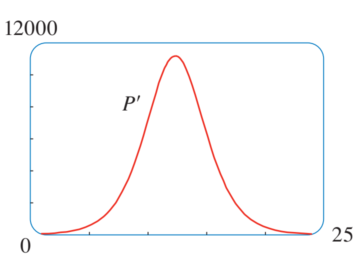


FIGURE 4

EXAMPLE C A population of honeybees raised in an apiary started with 50 bees at time $t = 0$ and was modeled by the function

$$P(t) = \frac{75{,}200}{1 + 1503e^{-0.5932t}}$$

where t is the time in weeks, $0 \leqslant t \leqslant 25$. Use a graph to estimate the time at which the bee population was growing fastest. Then use derivatives to give a more accurate estimate.

SOLUTION The population grows fastest when the population curve $y = P(t)$ has the steepest tangent line. From the graph of P in Figure 3, we estimate that the steepest tangent occurs when $t \approx 12$, so the bee population was growing most rapidly after about 12 weeks.

For a better estimate we calculate the derivative $P'(t)$, which is the rate of increase of the bee population:

$$P'(t) = -\frac{67{,}046{,}785.92e^{-0.5932t}}{(1 + 1503e^{-0.5932t})^2}$$

We graph P' in Figure 4 and observe that P' has its maximum value when $t \approx 12.3$.

To get a still better estimate we note that f' has its maximum value when f' changes from increasing to decreasing. This happens when f changes from concave upward to concave downward, that is, when f has an inflection point. So we ask a CAS to compute the second derivative:

$$P''(t) \approx \frac{119555093144e^{-1.1864t}}{(1 + 1503e^{-0.5932t})^3} - \frac{39772153e^{-0.5932t}}{(1 + 1503e^{-0.5932t})^2}$$

We could plot this function to see where it changes from positive to negative, but instead let's have the CAS solve the equation $P''(t) = 0$. It gives the answer $t \approx 12.3318$. ■

EXAMPLE D Investigate the family of functions given by $f(x) = cx + \sin x$. What features do the members of this family have in common? How do they differ?

SOLUTION The derivative is $f'(x) = c + \cos x$. If $c > 1$, then $f'(x) > 0$ for all x (since $\cos x \geqslant -1$), so f is always increasing. If $c = 1$, then $f'(x) = 0$ when x is an odd multiple of π, but f just has horizontal tangents there and is still an increasing function. Similarly, if $c \leqslant -1$, then f is always decreasing. If $-1 < c < 1$, then the equation $c + \cos x = 0$ has infinitely many solutions $[x = 2n\pi \pm \cos^{-1}(-c)]$ and f has infinitely many minima and maxima.

The second derivative is $f''(x) = -\sin x$, which is negative when $0 < x < \pi$ and, in general, when $2n\pi < x < (2n + 1)\pi$, where n is any integer. Thus, *all* members of the family are concave downward on $(0, \pi)$, $(2\pi, 3\pi)$, . . . and concave upward on $(\pi, 2\pi)$, $(3\pi, 4\pi)$, ... This is illustrated by several members of the family in Figure 5.

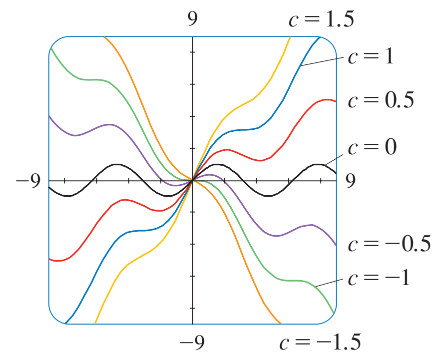


FIGURE 5 ■