EXAMPLE A Find $\int \sqrt{1 + x^2}\, x^5\, dx$.

SOLUTION An appropriate substitution becomes more obvious if we factor x^5 as $x^4 \cdot x$. Let $u = 1 + x^2$. Then $du = 2x\, dx$, so $x\, dx = du/2$. Also $x^2 = u - 1$, so $x^4 = (u - 1)^2$:

$$\int \sqrt{1 + x^2}\, x^5\, dx = \int \sqrt{1 + x^2}\, x^4 \cdot x\, dx$$

$$= \int \sqrt{u}\,(u - 1)^2\, \frac{du}{2} = \tfrac{1}{2}\int \sqrt{u}\,(u^2 - 2u + 1)\, du$$

$$= \tfrac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\, du$$

$$= \tfrac{1}{2}\left(\tfrac{2}{7}u^{7/2} - 2 \cdot \tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right) + C$$

$$= \tfrac{1}{7}(1 + x^2)^{7/2} - \tfrac{2}{5}(1 + x^2)^{5/2} + \tfrac{1}{3}(1 + x^2)^{3/2} + C \qquad \blacksquare$$

EXAMPLE B Evaluate $\int_0^4 \sqrt{2x + 1}\, dx$ using (6).

SOLUTION Using the substitution from Solution 1 of Example 2, we have $u = 2x + 1$ and $dx = du/2$. To find the new limits of integration we note that

when $x = 0$, $u = 2(0) + 1 = 1$ and when $x = 4$, $u = 2(4) + 1 = 9$

Therefore
$$\int_0^4 \sqrt{2x + 1}\, dx = \int_1^9 \tfrac{1}{2}\sqrt{u}\, du$$

$$= \tfrac{1}{2} \cdot \tfrac{2}{3}u^{3/2}\Big]_1^9$$

$$= \tfrac{1}{3}(9^{3/2} - 1^{3/2}) = \tfrac{26}{3}$$

▪ The geometric interpretation of Example B is shown in Figure 1. The substitution $u = 2x + 1$ stretches the interval $[0, 4]$ by a factor of 2 and translates it to the right by 1 unit. The Substitution Rule shows that the two areas are equal.

Observe that when using (6) we do not return to the variable x after integrating. We simply evaluate the expression in u between the appropriate values of u. ▪
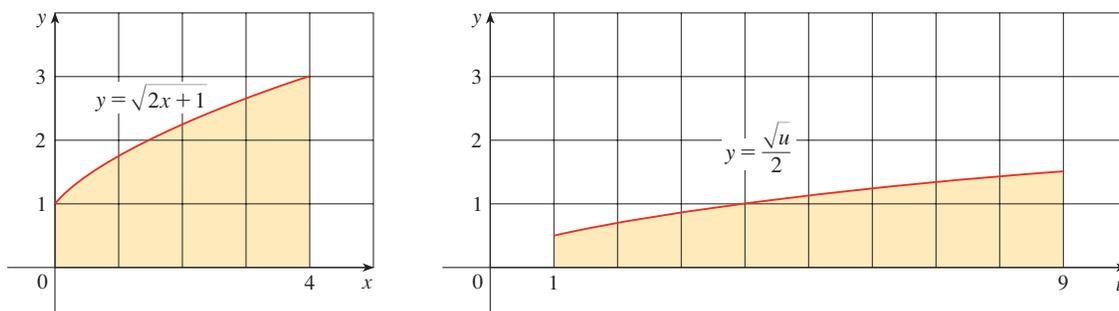

FIGURE I