3.5 INVERSE TRIGONOMETRIC FUNCTIONS

A Click here for answers.

S Click here for solutions.

1–16 ▪ Find the exact value of the expression.

1. $\sin^{-1}(0.5)$

2. $\csc^{-1}\sqrt{2}$

3. $\cot^{-1}(\sqrt{3})$

4. $\sec^{-1} 2$

5. $\arcsin\left(\sin\dfrac{5\pi}{4}\right)$

6. $\sin\left(2\sin^{-1}\left(\frac{3}{5}\right)\right)$

7. $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$

8. $\sin^{-1}\left(-\dfrac{1}{\sqrt{2}}\right)$

9. $\sin(\sin^{-1} 0.7)$

10. $\tan^{-1}\left(\tan\dfrac{4\pi}{3}\right)$

11. $\sin^{-1}(\sin 1)$

12. $\tan(\cos^{-1} 0.5)$

13. $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$

14. $\sec(\arctan 2)$

15. $\cos\left(2\sin^{-1}\left(\frac{5}{13}\right)\right)$

16. $\sin\left[\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2}{3}\right)\right]$

17. It is a fact that $\cos(\sin^{-1}x) = \sqrt{1 - x^2}$ for $-1 \leqslant x \leqslant 1$. Find a similar simplification of $\sin(2\cos^{-1}x)$.

18–35 ▪ Find the derivative of the function. Simplify where possible.

18. $f(x) = \sin^{-1}(2x - 1)$

19. $g(x) = \tan^{-1}(x^3)$

20. $h(x) = (\arcsin x)\ln x$

21. $f(t) = (\cos^{-1}t)/t$

22. $F(t) = \sqrt{1 - t^2} + \sin^{-1}t$

23. $G(t) = \cos^{-1}\sqrt{2t - 1}$

24. $g(x) = \tan^{-1}(x^3)$

25. $G(x) = \sin^{-1}(x/a),\ a > 0$

26. $F(x) = \tan^{-1}(x/a)$

27. $y = (\sin^{-1}x)^2$

28. $y = \sin^{-1}(x^2)$

29. $y = \tan^{-1}(e^x)$

30. $f(x) = (\arctan x)\ln x$

31. $g(t) = \sin^{-1}(4/t)$

32. $y = x^2\cot^{-1}(3x)$

33. $y = \tan^{-1}(\sin x)$

34. $y = \sin^{-1}\left(\dfrac{\cos x}{1 + \sin x}\right)$

35. $y = (\tan^{-1}x)^{-1}$

36–41 ▪ Find the derivative of the function. Find the domains of the function and its derivative.

36. $f(x) = \cos^{-1}(\sin^{-1}x)$

37. $g(x) = \sin^{-1}(3x + 1)$

38. $F(x) = \sqrt{\sin^{-1}(2/x)}$

39. $S(x) = \sin^{-1}(\tan^{-1}x)$

40. $R(t) = \arcsin(2^t)$

41. $U(t) = 2^{\arctan t}$

42. If $h(x) = (3\tan^{-1}x)^4$, find $h'(3)$.

43–48 ▪ Find the limit.

43. $\displaystyle\lim_{x\to\infty}(\tan^{-1}x)^2$

44. $\displaystyle\lim_{x\to 1^-}\frac{\arcsin x}{\tan(\pi x/2)}$

45. $\displaystyle\lim_{x\to\infty}\frac{\tan^{-1}x}{x}$

46. $\displaystyle\lim_{x\to\infty}\sin^{-1}\left(\frac{x + 1}{2x + 1}\right)$

47. $\displaystyle\lim_{x\to\infty}\tan^{-1}(x^2)$

48. $\displaystyle\lim_{x\to\infty}\tan^{-1}(x - x^2)$

3.5 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $\frac{\pi}{6}$

2. $\frac{\pi}{4}$

3. $\frac{5\pi}{6}$

4. $\frac{\pi}{3}$

5. $-\frac{\pi}{4}$

6. $\frac{24}{25}$

7. $\frac{\pi}{6}$

8. $-\frac{\pi}{4}$

9. 0.7

10. $\frac{\pi}{3}$

11. 1

12. $\sqrt{3}$

13. $\frac{3}{5}$

14. $\sqrt{5}$

15. $\frac{119}{169}$

16. $\frac{1}{9}\left(\sqrt{5}+4\sqrt{2}\right)$

17. $2x\sqrt{1-x^2}$

18. $f'(x) = \dfrac{1}{\sqrt{x-x^2}}$

19. $g'(x) = \dfrac{3x^2}{1+x^6}$

20. $h'(x) = \dfrac{\ln x}{\sqrt{1-x^2}} + \dfrac{\arcsin x}{x}$

21. $f'(t) = -\dfrac{\cos^{-1} t}{t^2} - \dfrac{1}{t\sqrt{1-t^2}}$

22. $F'(t) = \dfrac{1-t}{\sqrt{1-t^2}}$

23. $G'(t) = -\dfrac{1}{\sqrt{2\left(-2t^2+3t-1\right)}}$

24. $g'(x) = \dfrac{3x^2}{1+x^6}$

25. $G'(x) = \dfrac{1}{\sqrt{a^2-x^2}}$

26. $F'(x) = \dfrac{a}{a^2+x^2}$

27. $y' = \dfrac{2\sin^{-1} x}{\sqrt{1-x^2}}$

28. $y' = \dfrac{2x}{\sqrt{1-x^4}}$

29. $y' = \dfrac{e^x}{1+e^{2x}}$

30. $f'(x) = \dfrac{\arctan x}{x} + \dfrac{\ln x}{1+x^2}$

31. $g'(t) = -\dfrac{4}{\sqrt{t^4-16t^2}}$

32. $y' = 2x\cot^{-1}(3x) - \dfrac{3x^2}{1+9x^2}$

33. $y' = \dfrac{\cos x}{1+\sin^2 x}$

34. $y' = \dfrac{-1}{\sqrt{2\sin x + 2\sin^2 x}}$

35. $y' = -\dfrac{1}{(1+x^2)\left(\tan^{-1} x\right)^2}$

36. $f'(x) = -\dfrac{1}{\sqrt{1-\left(\sin^{-1} x\right)^2}} \cdot \dfrac{1}{\sqrt{1-x^2}}$, $[-\sin 1, \sin 1]$, $(-\sin 1, \sin 1)$

37. $g'(x) = \dfrac{3}{\sqrt{-9x^2-6x}}$, $\left[-\frac{2}{3}, 0\right]$, $\left(-\frac{2}{3}, 0\right)$

38. $F'(x) = -\dfrac{1}{x^2\sqrt{\sin^{-1}(2/x)}\sqrt{1-4/x^2}}$, $[2, \infty)$, $(2, \infty)$

39. $S'(x) = \left[\sqrt{1-\left(\tan^{-1} x\right)^2}\left(1+x^2\right)\right]^{-1}$, $[-\tan 1, \tan 1]$, $(-\tan 1, \tan 1)$

40. $R'(t) = \dfrac{2^t \ln 2}{\sqrt{1-4^t}}$, $(-\infty, 0]$, $(-\infty, 0)$

41. $U'(t) = 2^{\arctan t}(\ln 2)/\left(1+t^2\right)$, $\mathbb{R}$, $\mathbb{R}$

42. $\frac{162}{5}\left(\tan^{-1} 3\right)^3$

43. $\dfrac{\pi^2}{4}$

44. 0

45. 0

46. $\dfrac{\pi}{6}$

47. $\dfrac{\pi}{2}$

48. $-\dfrac{\pi}{2}$

3.5 SOLUTIONS

E Click here for exercises.

1. $\sin^{-1}(0.5) = \frac{\pi}{6}$ because $\sin\frac{\pi}{6} = 0.5$.

2. $\csc^{-1}\sqrt{2} = \frac{\pi}{4}$ because $\csc\frac{\pi}{4} = \sqrt{2}$.

3. $\cot^{-1}\left(-\sqrt{3}\right) = \frac{5\pi}{6}$ because $\cot\frac{5\pi}{6} = -\sqrt{3}$.

4. $\sec^{-1} 2 = \frac{\pi}{3}$ because $\sec\frac{\pi}{3} = 2$.

5. $\arcsin\left(\sin\frac{5\pi}{4}\right) = \arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

6. Let $\theta = \sin^{-1}\left(\frac{3}{5}\right)$. Then $\sin\theta = \frac{3}{5}$

$\Rightarrow \quad \cos\theta = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \frac{4}{5}$, so

$\sin\left(2\sin^{-1}\left(\frac{3}{5}\right)\right) = \sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$.

7. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$ because $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.

8. $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$ because $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

9. $\sin\left(\sin^{-1}(0.7)\right) = 0.7$ since 0.7 is in $[\ \ 1, 1]$.

10. $\tan^{-1}\left(\tan\frac{4\pi}{3}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$ since $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

11. $\sin^{-1}(\sin 1) = 1$ since $-\frac{\pi}{2} \le 1 \le \frac{\pi}{2}$.

12. $\tan\left(\cos^{-1} 0.5\right) = \tan\frac{\pi}{3} = \sqrt{3}$

13. Let $\theta = \cos^{-1}\left(\frac{4}{5}\right)$, so $\cos\theta = \frac{4}{5}$. Then

$\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right) = \sin\theta = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}$.

14. Let $\theta = \arctan 2$, so $\tan\theta = 2 \quad \Rightarrow$

$\sec^2\theta = 1 + \tan^2\theta = 1 + 4 = 5 \quad \Rightarrow \quad \sec\theta = \sqrt{5} \quad \Rightarrow$

$\sec(\arctan 2) = \sec\theta = \sqrt{5}$.

15. Let $\theta = \sin^{-1}\left(\frac{5}{13}\right)$. Then $\sin\theta = \frac{5}{13}$, so

$$\cos\left(2\sin^{-1}\left(\frac{5}{13}\right)\right) = \cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{5}{13}\right)^2 = \frac{119}{169}.$$

16. Let $x = \sin^{-1}\left(\frac{1}{3}\right)$ and $y = \sin^{-1}\left(\frac{2}{3}\right)$. Then

$\sin x = \frac{1}{3}$, $\cos x = \sqrt{1 - \left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$, $\sin y = \frac{2}{3}$,

$\cos y = \sqrt{1 - \left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}$, so

$$\begin{aligned} \sin\left(\sin^{-1}\left(\tfrac{1}{3}\right) + \sin^{-1}\left(\tfrac{2}{3}\right)\right) &= \sin(x+y) \\ &= \sin x\cos y + \cos x\sin y \\ &= \tfrac{1}{3}\left(\tfrac{\sqrt{5}}{3}\right) + \tfrac{2\sqrt{2}}{3}\left(\tfrac{2}{3}\right) \\ &= \tfrac{1}{9}\left(\sqrt{5} + 4\sqrt{2}\right) \end{aligned}$$

17. Let $y = \cos^{-1} x$. Then $\cos y = x$

$\Rightarrow \quad \sin y = \sqrt{1 - x^2}$ since $0 \le y \le \pi$. So

$\sin\left(2\cos^{-1} x\right) = \sin 2y = 2\sin y\cos y = 2x\sqrt{1 - x^2}$.

18. $f(x) = \sin^{-1}(2x - 1) \quad \Rightarrow$

$$f'(x) = \frac{1}{\sqrt{1 - (2x-1)^2}}(2) = \frac{1}{\sqrt{x - x^2}}$$

19. $g(x) = \tan^{-1}\left(x^3\right) \quad \Rightarrow$

$$g'(x) = \frac{1}{1 + \left(x^3\right)^2}\left(3x^2\right) = \frac{3x^2}{1 + x^6}$$

20. $h(x) = (\arcsin x)\ln x \quad \Rightarrow$

$$h'(x) = \frac{\ln x}{\sqrt{1 - x^2}} + \frac{\arcsin x}{x}$$

21. $f(t) = \dfrac{\cos^{-1} t}{t} \quad \Rightarrow$

$$\begin{aligned} f'(t) &= \frac{t\left(-1/\sqrt{1 - t^2}\right) - \cos^{-1} t}{t^2} \\ &= -\frac{\cos^{-1} t}{t^2} - \frac{1}{t\sqrt{1 - t^2}} \end{aligned}$$

22. $F(t) = \sqrt{1 - t^2} + \sin^{-1} t \quad \Rightarrow$

$$F'(t) = \frac{-2t}{2\sqrt{1 - t^2}} + \frac{1}{\sqrt{1 - t^2}} = \frac{1 - t}{\sqrt{1 - t^2}}$$

23. $G(t) = \cos^{-1}\sqrt{2t - 1} \quad \Rightarrow$

$$\begin{aligned} G'(t) &= -\frac{1}{\sqrt{1 - (2t-1)}}\frac{2}{2\sqrt{2t - 1}} \\ &= -\frac{1}{\sqrt{2\left(-2t^2 + 3t - 1\right)}} \end{aligned}$$

24. $g(x) = \tan^{-1}\left(x^3\right) \quad \Rightarrow$

$$g'(x) = \frac{1}{1 + \left(x^3\right)^2}\left(3x^2\right) = \frac{3x^2}{1 + x^6}$$

25. $G(x) = \sin^{-1}(x/a) \quad \Rightarrow$

$$G'(x) = \frac{1/a}{\sqrt{1 - (x/a)^2}} = \frac{1}{a\sqrt{1 - x^2/a^2}} = \frac{1}{\sqrt{a^2 - x^2}}$$

26. $F(x) = \tan^{-1}(x/a) \quad \Rightarrow$

$$F'(x) = \frac{1}{1 + (x/a)^2}\cdot\frac{1}{a} = \frac{a}{a^2 + x^2}$$

27. $y = \left(\sin^{-1} x\right)^2 \quad \Rightarrow$

$$y' = 2\left(\sin^{-1} x\right)\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{2\sin^{-1} x}{\sqrt{1 - x^2}}$$

28. $y = \sin^{-1}\left(x^2\right) \quad \Rightarrow$

$$y' = \frac{1}{\sqrt{1 - \left(x^2\right)^2}}\frac{d}{dx}\left(x^2\right) = \frac{2x}{\sqrt{1 - x^4}}$$

29. $y = \tan^{-1}\left(e^x\right) \quad \Rightarrow \quad y' = \dfrac{1}{1 + \left(e^x\right)^2}\dfrac{d}{dx}\left(e^x\right) = \dfrac{e^x}{1 + e^{2x}}$

30. $f(x) = (\arctan x)\ln x \quad\Rightarrow$

$$f'(x) = \arctan x \cdot \frac{1}{x} + \ln x \cdot \frac{1}{1+x^2} = \frac{\arctan x}{x} + \frac{\ln x}{1+x^2}$$

31. $g(t) = \sin^{-1}\left(\frac{4}{t}\right) \quad\Rightarrow$

$$g'(t) = \frac{1}{\sqrt{1-(4/t)^2}}\left(-\frac{4}{t^2}\right) = -\frac{4}{\sqrt{t^4-16t^2}}$$

32. $y = x^2\cot^{-1}(3x) \quad\Rightarrow$

$$y' = 2x\cot^{-1}(3x) + x^2\left[-\frac{1}{1+(3x)^2}\right](3) =$$

$$2x\cot^{-1}(3x) - \frac{3x^2}{1+9x^2}$$

33. $y = \tan^{-1}(\sin x) \quad\Rightarrow\quad y' = \dfrac{\cos x}{1+\sin^2 x}$

34. $y = \sin^{-1}\left(\dfrac{\cos x}{1+\sin x}\right) \quad\Rightarrow$

$$\begin{aligned} y' &= \frac{1}{\sqrt{1-[\cos x/(1+\sin x)]^2}} \cdot \frac{-\sin x(1+\sin x) - \cos^2 x}{(1+\sin x)^2} \\ &= \frac{1}{\sqrt{(1+2\sin x+\sin^2 x-\cos^2 x)/(1+\sin x)^2}} \cdot \frac{-(1+\sin x)}{(1+\sin x)^2} \\ &= \frac{-1}{\sqrt{2\sin x+2\sin^2 x}} \end{aligned}$$

35. $y = \left(\tan^{-1}x\right)^{-1} \quad\Rightarrow$

$$y' = -\left(\tan^{-1}x\right)^{-2}\left(\frac{1}{1+x^2}\right) = -\frac{1}{(1+x^2)\left(\tan^{-1}x\right)^2}$$

36. $f(x) = \cos^{-1}\left(\sin^{-1}x\right) \quad\Rightarrow$

$$f'(x) = -\frac{1}{\sqrt{1-\left(\sin^{-1}x\right)^2}} \cdot \frac{1}{\sqrt{1-x^2}},$$

$$\begin{aligned} \text{Dom}(f) &= \left\{x \mid -1 \le \sin^{-1}x \le 1\right\} \\ &= \{x \mid \sin(-1) \le x \le \sin 1\} \\ &= [-\sin 1, \sin 1] \end{aligned}$$

$\text{Dom}(f') = \left\{x \mid -1 < \sin^{-1}x < 1\right\} = (-\sin 1, \sin 1)$

37. $g(x) = \sin^{-1}(3x+1) \quad\Rightarrow$

$$g'(x) = \frac{3}{\sqrt{1-(3x+1)^2}} = \frac{3}{\sqrt{-9x^2-6x}},$$

$$\begin{aligned} \text{Dom}(g) &= \{x \mid -1 \le 3x+1 \le 1\} \\ &= \left\{x \mid -\tfrac{2}{3} \le x \le 0\right\} = \left[-\tfrac{2}{3}, 0\right] \end{aligned}$$

$\text{Dom}(g') = \{x \mid -1 < 3x+1 < 1\} = \left(-\tfrac{2}{3}, 0\right)$

38. $F(x) = \sqrt{\sin^{-1}(2/x)} \quad\Rightarrow$

$$\begin{aligned} F'(x) &= \frac{1}{2\sqrt{\sin^{-1}(2/x)}} \cdot \frac{1}{\sqrt{1-(2/x)^2}}\left[-\frac{2}{x^2}\right] \\ &= -\frac{1}{x^2\sqrt{\sin^{-1}(2/x)}\sqrt{1-4/x^2}} \end{aligned}$$

$$\begin{aligned} \text{Dom}(F) &= \left\{x \mid -1 \le 2/x \le 1 \text{ and } \sin^{-1}(2/x) \ge 0\right\} \\ &= \{x \mid 0 < 2/x \le 1\} = \{x \mid x \ge 2\} = [2, \infty) \end{aligned}$$

$\text{Dom}(F') = \{x \mid x > 2\} = (2, \infty)$

39. $S(x) = \sin^{-1}\left(\tan^{-1}x\right) \quad\Rightarrow$

$$S'(x) = \left[\sqrt{1-\left(\tan^{-1}x\right)^2}\left(1+x^2\right)\right]^{-1}$$

$$\begin{aligned} \text{Dom}(S) &= \left\{x \mid -1 \le \tan^{-1}x \le 1\right\} \\ &= \{x \mid \tan(-1) \le x \le \tan 1\} = [-\tan 1, \tan 1] \end{aligned}$$

$\text{Dom}(S') = \left\{x \mid -1 < \tan^{-1}x < 1\right\} = (-\tan 1, \tan 1)$

40. $R(t) = \arcsin 2^t \quad\Rightarrow$

$$R'(t) = \frac{1}{\sqrt{1-(2^t)^2}}\left(2^t\ln 2\right) = \frac{2^t\ln 2}{\sqrt{1-4^t}},$$

$\text{Dom}(R) = \left\{t \mid -1 \le 2^t \le 1\right\} = \{t \mid t \le 0\} = (-\infty, 0]$,

$\text{Dom}(R') = (-\infty, 0)$

41. $U(t) = 2^{\arctan t} \quad\Rightarrow\quad U'(t) = 2^{\arctan t}(\ln 2)/\left(1+t^2\right)$,

$\text{Dom}(U) = \text{Dom}(U') = \mathbb{R}$

42. $h(x) = \left(3\tan^{-1}x\right)^4 \quad\Rightarrow$

$$h'(x) = 4\left(3\tan^{-1}x\right)^3\left(\frac{3}{1+x^2}\right) \quad\Rightarrow$$

$$h'(3) = 4\left(3\tan^{-1}3\right)^3\left(\tfrac{3}{10}\right) = \tfrac{162}{5}\left(\tan^{-1}3\right)^3$$

43. $\displaystyle\lim_{x\to\infty}\left(\tan^{-1}x\right)^2 = \left(\lim_{x\to\infty}\tan^{-1}x\right)^2 = \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}$

44. $\displaystyle\lim_{x\to 1^-}\frac{\arcsin x}{\tan(\pi x/2)} = 0$ since $\displaystyle\lim_{x\to 1^-}\arcsin x = \tfrac{\pi}{2}$ and $\displaystyle\lim_{x\to 1^-}\tan(\pi x/2) = \infty$.

45. $\displaystyle\lim_{x\to\infty}\frac{\tan^{-1}x}{x} = 0$ since $\displaystyle\lim_{x\to\infty}\tan^{-1}x = \tfrac{\pi}{2}$ and $\displaystyle\lim_{x\to\infty} x = \infty$.

Or: Note that $0 < \dfrac{\tan^{-1}x}{x} < \dfrac{\pi/2}{x}$ and use the Squeeze Theorem.

46. $\displaystyle\lim_{x\to\infty}\sin^{-1}\left(\frac{x+1}{2x+1}\right) = \sin^{-1}\left(\lim_{x\to\infty}\frac{x+1}{2x+1}\right) = \sin^{-1}\tfrac{1}{2} = \tfrac{\pi}{6}$

47. $\displaystyle\lim_{x\to\infty}\tan^{-1}\left(x^2\right) = \tfrac{\pi}{2}$ since $x^2 \to \infty$ as $x \to \infty$.

48. $\displaystyle\lim_{x\to\infty}\tan^{-1}\left(x-x^2\right) = -\tfrac{\pi}{2}$ since $x - x^2 = x(1-x) \to -\infty$ as $x \to \infty$.