4.1 MAXIMUM AND MINIMUM VALUES

A Click here for answers.

1–4 ▪ Sketch the graph of a function f that is continuous on $[0, 3]$ and has the given properties.

1. Absolute maximum at 0, absolute minimum at 3, local minimum at 1, local maximum at 2

2. Absolute maximum at 1, absolute minimum at 2

3. 2 is a critical number, but f has no local maximum or minimum

4. Absolute minimum at 0, absolute maximum at 2, local maxima at 1 and 2, local minimum at 1.5

5–18 ▪ Sketch the graph of f by hand and use your sketch to find the absolute and local maximum and minimum values of f. (Use the graphs and transformations of Section 1.2.)

5. $f(x) = 1 + 2x; \quad x \geqslant -1$

6. $f(x) = 4x - 1; \quad x \leqslant 8$

7. $f(x) = 1 - x^2; \quad 0 < x < 1$

8. $f(x) = 1 - x^2; \quad 0 < x \leqslant 1$

9. $f(x) = 1 - x^2; \quad 0 \leqslant x < 1$

10. $f(x) = 1 - x^2; \quad 0 \leqslant x \leqslant 1$

11. $f(x) = 1 - x^2; \quad -2 \leqslant x \leqslant 1$

12. $f(x) = |x|; \quad -2 \leqslant x \leqslant 1$

13. $f(x) = |4x - 1|; \quad 0 \leqslant x \leqslant 2$

14. $f(\theta) = \cos(\theta/2); \quad -\pi < \theta < \pi$

15. $f(\theta) = \sec\theta; \quad -\pi/2 < \theta \leqslant \pi/3$

16. $f(x) = x^5$

17. $f(x) = 2 - x^4$

18. $f(x) = 1 - e^{-x}, \quad x \geqslant 0$

19–34 ▪ Find the critical numbers of the function.

19. $f(x) = 2x - 3x^2$

20. $f(x) = 5 + 8x$

21. $f(x) = x^3 - 3x + 1$

22. $f(t) = t^3 + 6t^2 + 3t - 1$

23. $g(x) = \sqrt[9]{x}$

24. $g(x) = |x + 1|$

25. $f(x) = 5 + 6x - 2x^3$

26. $f(t) = 2t^3 + 3t^2 + 6t + 4$

S Click here for solutions.

27. $f(x) = 4x^3 - 9x^2 - 12x + 3$

28. $s(t) = 2t^3 + 3t^2 - 6t + 4$

29. $s(t) = t^4 + 4t^3 + 2t^2$

30. $f(r) = \dfrac{r}{r^2 + 1}$

31. $f(\theta) = \sin^2(2\theta)$

32. $g(\theta) = \theta + \sin\theta$

33. $V(x) = x\sqrt{x - 2}$

34. $T(x) = x^2(2x - 1)^{2/3}$

35–45 ▪ Find the absolute maximum and absolute minimum values of f on the given interval.

35. $f(x) = x^2 - 2x + 2, \quad [0, 3]$

36. $f(x) = 1 - 2x - x^2, \quad [-4, 1]$

37. $f(x) = x^3 - 12x + 1, \quad [-3, 5]$

38. $f(x) = 4x^3 - 15x^2 + 12x + 7, \quad [0, 3]$

39. $f(x) = 2x^3 + 3x^2 + 4, \quad [-2, 1]$

40. $f(x) = 18x + 15x^2 - 4x^3, \quad [-3, 4]$

41. $f(x) = x^4 - 4x^2 + 2, \quad [-3, 2]$

42. $f(x) = 3x^5 - 5x^3 - 1, \quad [-2, 2]$

43. $f(x) = x^2 + 2/x, \quad [\tfrac{1}{2}, 2]$

44. $f(x) = \sqrt{9 - x^2}, \quad [-1, 2]$

45. $f(x) = \dfrac{x}{x + 1}, \quad [1, 2]$

46. Use a graph to estimate the critical numbers of $f(x) = x^4 - 3x^2 + x$ correct to one decimal place.

47. (a) Use a graph to estimate the absolute maximum and minimum values of $f(x) = x^4 - 3x^3 + 3x^2 - x$ on the interval $[0, 2]$, correct to two decimal places.
(b) Use calculus to find the exact maximum and minimum values.

48. Show that 0 is a critical number of the function $f(x) = x^5$ but f has neither a local maximum nor a local minimum at 0.

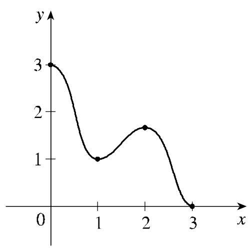
49. Sketch the graph of a function on $[0, 1]$ that is discontinuous and has both an absolute maximum and an absolute minimum.

4.1 ANSWERS

E Click here for exercises.

S Click here for solutions.

1.



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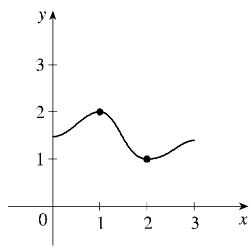

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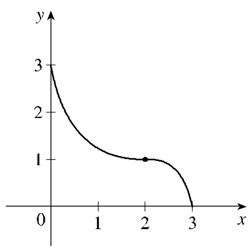

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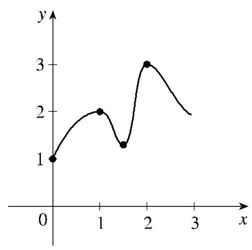


5. Abs. min. $f(-1) = -1$ **6.** Abs. max. $f(8) = 31$

7. None **8.** Abs. min. $f(1) = 0$ **9.** Abs. max. $f(0) = 1$

10. Abs. max. $f(0) = 1$; abs. min. $f(1) = 0$

11. Abs. and loc. max. $f(0) = 1$; abs. min. $f(-2) = -3$

12. Abs. max. $f(-2) = 2$; abs. and loc. min $f(0) = 0$

13. Abs. max. $f(2) = 7$; abs. and loc. min. $f\left(\frac{1}{4}\right) = 0$

14. Abs. and loc. max. $f(0) = 1$

15. Abs. and loc. min. $f(0) = 1$ **16.** None

17. Abs. and loc. max. $f(0) = 2$ **18.** Abs. min. $f(0) = 0$

19. $\frac{1}{3}$ **20.** None **21.** ± 1 **22.** $-2 \pm \sqrt{3}$

23. 0 **24.** -1 **25.** ± 1 **26.** None

27. $-\frac{1}{2}, 2$ **28.** $\left(-1 \pm \sqrt{5}\right)/2$ **29.** $0, \left(-3 \pm \sqrt{5}\right)/2$

30. ± 1 **31.** $n\pi/4$, n an integer

32. $(2n+1)\pi$, n an integer **33.** 2 **34.** $0, \frac{1}{2}, \frac{3}{8}$

35. Abs. max. $f(3) = 5$; abs. min. $f(1) = 1$

36. Abs. max. $f(-1) = 2$; abs. min. $f(-4) = -7$

37. Abs. max. $f(5) = 66$; abs. min. $f(2) = -15$

38. Abs. max. $f(3) = 16$; abs. min. $f(2) = 3$

39. Abs. max. $f(1) = 9$; abs.min. $f(-2) = 0$

40. Abs. max. $f(-3) = 189$; abs.min. $f\left(-\frac{1}{2}\right) = -\frac{19}{4}$

41. Abs. max. $f(-3) = 47$; abs. min. $f\left(\pm\sqrt{2}\right) = -2$

42. Abs. max. $f(2) = 55$; abs. min. $f(-2) = -57$

43. Abs. max. $f(2) = 5$; abs. min. $f(1) = 3$

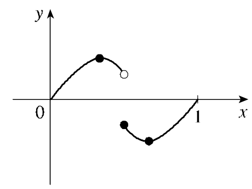
44. Abs. max. $f(0) = 3$; abs. min. $f(2) = \sqrt{5}$

45. Abs. max. $f(2) = \frac{2}{3}$; abs. min. $f(1) = \frac{1}{2}$

46. $-1.3, 0.2, 1.1$

47. (a) $2, -0.11$ (b) $2, -\frac{27}{256}$

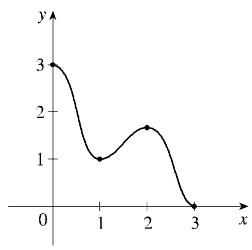
49.



4.1 SOLUTIONS

E Click here for exercises.

1.



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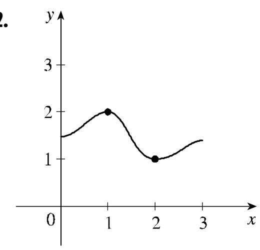

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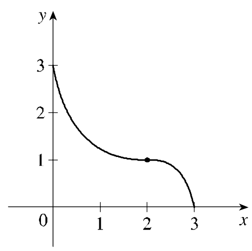

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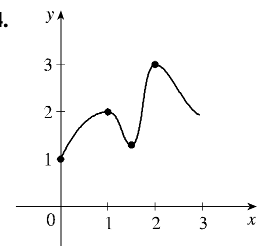

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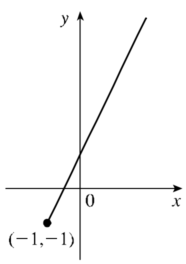
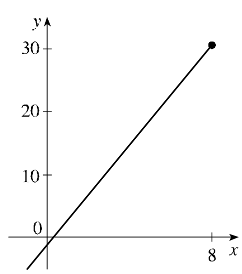


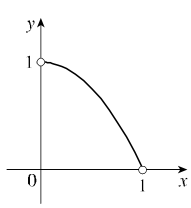
$f(x) = 1 + 2x$, $x \geq -1$. Absolute minimum $f(-1) = -1$; no local minimum. No local or absolute maximum.

6.



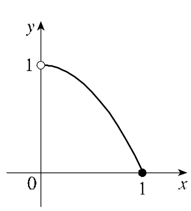
$f(x) = 4x - 1$, $x \leq 8$. Absolute maximum $f(8) = 31$; no local maximum. No local or absolute minimum.

7.



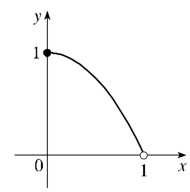
$f(x) = 1 - x^2$, $0 < x < 1$. No maximum or minimum.

8.



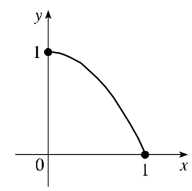
$f(x) = 1 - x^2$, $0 < x \leq 1$. Absolute minimum $f(1) = 0$; no local minimum. No absolute or local maximum.

9.



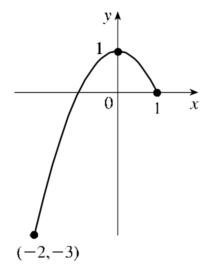
$f(x) = 1 - x^2$, $0 \leq x < 1$. Absolute maximum $f(0) = 1$; no local maximum. No absolute or local minimum.

10.



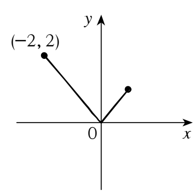
$f(x) = 1 - x^2$, $0 \leq x \leq 1$. Absolute maximum $f(0) = 1$; no local maximum. Absolute minimum $f(1) = 0$; no local minimum.

11.



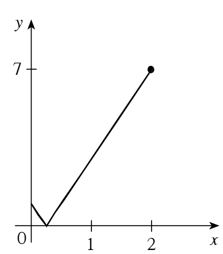
$f(x) = 1 - x^2$, $-2 \leq x \leq 1$. Absolute and local maximum $f(0) = 1$. Absolute minimum $f(-2) = -3$; no local minimum.

12.



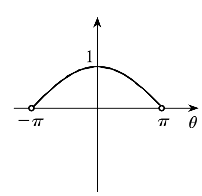
$f(x) = |x|$, $-2 \leq x \leq 1$. Absolute maximum $f(-2) = 2$; no local maximum. Absolute and local minimum $f(0) = 0$.

13.



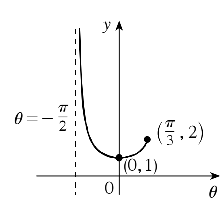
$f(x) = |4x - 1|$, $0 \leq x \leq 2$. Absolute maximum $f(2) = 7$; no local maximum. Absolute and local minimum $f\left(\frac{1}{4}\right) = 0$ since $|4x - 1| \geq 0$ for all x.

14.

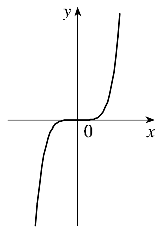


$f(\theta) = \cos(\theta/2)$, $-\pi < \theta < \pi$. Local and absolute maximum $f(0) = 1$. No local or absolute minimum.

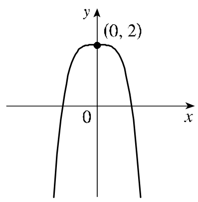
15.



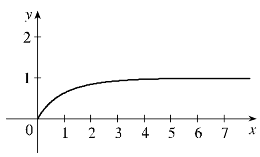
$f(\theta) = \sec\theta$, $-\frac{\pi}{2} < \theta \leq \frac{\pi}{3}$. Local and absolute minimum $f(0) = 1$. No absolute or local maximum.

16. 

$f(x) = x^5$. No maximum or minimum.

17. 

$f(x) = 2 - x^4$. Local and absolute maximum $f(0) = 2$. No local or absolute minimum.

18. 

$f(x) = 1 - e^{-x}$, $x \geq 0$. Absolute minimum $f(0) = 0$; no local minimum. No absolute or local maximum.

19. $f(x) = 2x - 3x^2 \quad \Rightarrow \quad f'(x) = 2 - 6x = 0 \quad \Leftrightarrow \quad x = \frac{1}{3}$. So the critical number is $\frac{1}{3}$.

20. $f(x) = 5 + 8x \quad \Rightarrow \quad f'(x) = 8 \neq 0$. No critical number.

21. $f(x) = x^3 - 3x + 1 \quad \Rightarrow \quad f'(x) = 3x^2 - 3 = 3\left(x^2 - 1\right) = 3(x+1)(x-1)$. So the critical numbers are ± 1.

22. $f(t) = t^3 + 6t^2 + 3t - 1 \quad \Rightarrow \quad f'(t) = 3t^2 + 12t + 3 = 3\left(t^2 + 4t + 1\right)$. By the quadratic formula, solutions are $t = \dfrac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$. Critical numbers are $t = -2 \pm \sqrt{3}$.

23. $g(x) = \sqrt[9]{x} = x^{1/9} \quad \Rightarrow \quad g'(x) = \frac{1}{9}x^{-8/9} = \dfrac{1}{9\sqrt[9]{x^8}} \neq 0$, but $g'(0)$ does not exist, so $x = 0$ is a critical number.

24. $g(x) = |x + 1| \quad \Rightarrow \quad g'(x) = 1$ if $x > -1$, $g'(x) = -1$ if $x < -1$, but $g'(-1)$ does not exist, so $x = -1$ is a critical number.

25. $f(x) = 5 + 6x - 2x^3 \quad \Rightarrow \quad f'(x) = 6 - 6x^2 = 6(1 + x)(1 - x)$. $f'(x) = 0 \quad \Rightarrow \quad x = \pm 1$, so ± 1 are the critical numbers.

26. $f(t) = 2t^3 + 3t^2 + 6t + 4 \quad \Rightarrow \quad f'(t) = 6t^2 + 6t + 6$. But $t^2 + t + 1 = 0$ has no real solution since $b^2 - 4ac = 1 - 4(1)(1) = -3 < 0$. No critical number.

27. $f(x) = 4x^3 - 9x^2 - 12x + 3 \quad \Rightarrow$

$$\begin{aligned} f'(x) &= 12x^2 - 18x - 12 \\ &= 6\left(2x^2 - 3x - 2\right) = 6(2x + 1)(x - 2) \end{aligned}$$

$f'(x) = 0 \quad \Rightarrow \quad x = -\frac{1}{2}, 2$; so the critical numbers are $x = -\frac{1}{2}, 2$.

28. $s(t) = 2t^3 + 3t^2 - 6t + 4 \quad \Rightarrow \quad s'(t) = 6t^2 + 6t - 6 = 6\left(t^2 + t - 1\right)$. By the quadratic formula, the critical numbers are $t = \left(-1 \pm \sqrt{5}\right)/2$.

29. $s(t) = t^4 + 4t^3 + 2t^2 \quad \Rightarrow \quad s'(t) = 4t^3 + 12t^2 + 4t = 4t\left(t^2 + 3t + 1\right) = 0$ when $t = 0$ or $t^2 + 3t + 1 = 0$. By the quadratic formula, the critical numbers are $t = 0, \frac{-3 \pm \sqrt{5}}{2}$.

30. $f(r) = \dfrac{r}{r^2 + 1} \quad \Rightarrow$

$$f'(r) = \frac{\left(r^2 + 1\right)1 - r(2r)}{\left(r^2 + 1\right)^2} = \frac{-r^2 + 1}{\left(r^2 + 1\right)^2} = 0 \quad \Leftrightarrow$$

$r^2 = 1 \quad \Leftrightarrow \quad r = \pm 1$, so these are the critical numbers. Note that $f'(r)$ always exists since $r^2 + 1 \neq 0$.

31. $f(\theta) = \sin^2(2\theta) \quad \Rightarrow$

$$\begin{aligned} f'(\theta) &= 2\sin(2\theta)\cos(2\theta)(2) \\ &= 2(2\sin 2\theta\cos 2\theta) = 2[\sin(2 \cdot 2\theta)] \\ &= 2\sin 4\theta = 0 \end{aligned}$$

$f(\theta) = 0 \quad \Leftrightarrow \quad \sin 4\theta = 0 \quad \Leftrightarrow \quad 4\theta = n\pi$, n an integer. So $\theta = n\pi/4$ are the critical numbers.

32. $g(\theta) = \theta + \sin\theta \quad \Rightarrow \quad g'(\theta) = 1 + \cos\theta = 0 \quad \Leftrightarrow \quad \cos\theta = -1$. The critical numbers are $\theta = \pi + 2n\pi = (2n + 1)\pi$, n an integer.

33. $V(x) = x\sqrt{x - 2} \quad \Rightarrow \quad V'(x) = \sqrt{x - 2} + \dfrac{x}{2\sqrt{x - 2}}$

$\Rightarrow \quad V'(-2)$ does not exist. For $x > 2$ [the domain of $V'(x)$], $V'(x) > 0$, so 2 is the only critical number.

34. $T(x) = x^2(2x - 1)^{2/3} \quad \Rightarrow$

$T'(x) = 2x(2x - 1)^{2/3} + x^2\left(\frac{2}{3}\right)(2x - 1)^{-1/3}(2)$. So $T'\left(\frac{1}{2}\right)$ does not exist.

$$\begin{aligned} T'(x) &= 2x(2x - 1)^{-1/3}\left(2x - 1 + \tfrac{2}{3}x\right) \\ &= 2x(2x - 1)^{-1/3}\left(\tfrac{8}{3}x - 1\right) = 0 \quad \Leftrightarrow \end{aligned}$$

$x = 0$ or $x = \frac{3}{8}$. So the critical numbers are $x = 0, \frac{3}{8}$, and $\frac{1}{2}$.

35. $f(x) = x^2 - 2x + 2$, $[0, 3]$. $f'(x) = 2x - 2 = 0 \quad \Leftrightarrow \quad x = 1$. $f(0) = 2$, $f(1) = 1$, $f(3) = 5$. So $f(3) = 5$ is the absolute maximum and $f(1) = 1$ is the absolute minimum.

36. $f(x) = 1 - 2x - x^2$, $[-4, 1]$. $f'(x) = -2 - 2x = 0 \quad \Leftrightarrow \quad x = -1$. $f(-4) = -7$, $f(-1) = 2$, $f(1) = -2$. So $f(-4) = -7$ is the absolute minimum, $f(-1) = 2$ is the absolute maximum.

37. $f(x) = x^3 - 12x + 1$, $[-3, 5]$. $f'(x) = 3x^2 - 12 = 3\left(x^2 - 4\right) = 3(x + 2)(x - 2) = 0 \quad \Leftrightarrow \quad x = \pm 2$. $f(-3) = 10$, $f(-2) = 17$, $f(2) = -15$, $f(5) = 66$. So $f(2) = -15$ is the absolute minimum and $f(5) = 66$ is the absolute maximum.

38. $f(x) = 4x^3 - 15x^2 + 12x + 7$, $[0, 3]$.
$f'(x) = 12x^2 - 30x + 12 = 6(2x-1)(x-2) = 0 \quad \Leftrightarrow$
$x = \frac{1}{2}, 2$. $f(0) = 7$, $f\left(\frac{1}{2}\right) = \frac{39}{4}$, $f(2) = 3$, $f(3) = 16$.
So $f(3) = 16$ is the absolute maximum and $f(2) = 3$ the absolute minimum.

39. $f(x) = 2x^3 + 3x^2 + 4$, $[-2, 1]$.
$f'(x) = 6x^2 + 6x = 6x(x+1) = 0 \quad \Leftrightarrow \quad x = -1, 0$.
$f(-2) = 0$, $f(-1) = 5$, $f(0) = 4$, $f(1) = 9$. So $f(1) = 9$ is the absolute maximum and $f(-2) = 0$ is the absolute minimum.

40. $f(x) = 18x + 15x^2 - 4x^3$, $[-3, 4]$.
$f'(x) = 18 + 30x - 12x^2 = 6(3-x)(1+2x) = 0 \quad \Leftrightarrow$
$x = 3, -\frac{1}{2}$. $f(-3) = 189$, $f\left(-\frac{1}{2}\right) = -\frac{19}{4}$, $f(3) = 81$,
$f(4) = 56$. So $f(-3) = 189$ is the absolute maximum and $f\left(-\frac{1}{2}\right) = -\frac{19}{4}$ is the absolute minimum.

41. $f(x) = x^4 - 4x^2 + 2$, $[-3, 2]$.
$f'(x) = 4x^3 - 8x = 4x\left(x^2 - 2\right) = 0 \quad \Leftrightarrow \quad x = 0, \pm\sqrt{2}$.
$f(-3) = 47$, $f\left(-\sqrt{2}\right) = -2$, $f(0) = 2$, $f\left(\sqrt{2}\right) = -2$,
$f(2) = 2$, so $f\left(\pm\sqrt{2}\right) = -2$ is the absolute minimum and $f(-3) = 47$ is the absolute maximum.

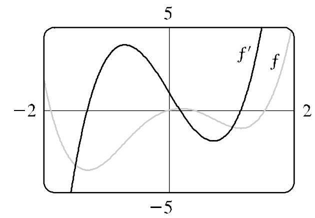
42. $f(x) = 3x^5 - 5x^3 - 1$, $[-2, 2]$.
$f'(x) = 15x^4 - 15x^2 = 15x^2(x+1)(x-1) = 0 \quad \Leftrightarrow$
$x = -1, 0, 1$. $f(-2) = -57$, $f(-1) = 1$, $f(0) = -1$,
$f(1) = -3$, $f(2) = 55$. So $f(-2) = -57$ is the absolute minimum and $f(2) = 55$ is the absolute maximum.

43. $f(x) = x^2 + \dfrac{2}{x}$, $\left[\frac{1}{2}, 2\right]$. $f'(x) = 2x - \dfrac{2}{x^2} = 2\dfrac{x^3 - 1}{x^2} = 0$
$\Leftrightarrow \quad x^3 - 1 = 0 \quad \Leftrightarrow \quad (x-1)\left(x^2 + x + 1\right) = 0$, but
$x^2 + x + 1 \neq 0$, so $x = 1$. The denominator is 0 at $x = 0$, but not in the desired interval. $f\left(\frac{1}{2}\right) = \frac{17}{4}$, $f(1) = 3$,
$f(2) = 5$. So $f(1) = 3$ is the absolute minimum and $f(2) = 5$ is the absolute maximum.

44. $f(x) = \sqrt{9 - x^2}$, $[-1, 2]$. $f'(x) = -x \big/ \sqrt{9 - x^2} = 0$
$\Leftrightarrow \quad x = 0$. $f(-1) = 2\sqrt{2}$, $f(0) = 3$, $f(2) = \sqrt{5}$. So $f(2) = \sqrt{5}$ is the absolute minimum and $f(0) = 3$ is the absolute maximum.

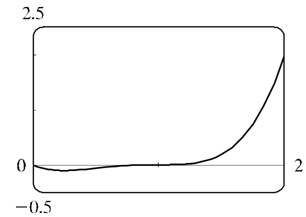
45. $f(x) = \dfrac{x}{x+1}$, $[1, 2]$.
$f'(x) = \dfrac{(x+1) - x}{(x+1)^2} = \dfrac{1}{(x+1)^2} \neq 0 \quad \Rightarrow \quad$ no critical number. $f(1) = \frac{1}{2}$ and $f(2) = \frac{2}{3}$, so $f(1) = \frac{1}{2}$ is the absolute minimum and $f(2) = \frac{2}{3}$ is the absolute maximum.

46.



We see that $f'(x) = 0$ at about $x = -1.3, 0.2$, and 1.1. Since f' exists everywhere, these are the only critical numbers.

47. (a)



From the graph, it appears that the absolute maximum value is $f(2) = 2$, and that the absolute minimum value is about $f(0.25) = -0.11$.

(b) $f(x) = x^4 - 3x^3 + 3x^2 - x \quad \Rightarrow$
$f'(x) = 4x^3 - 9x^2 + 6x - 1 = (4x-1)(x-1)^2$.
So $f'(x) = 0 \quad \Rightarrow \quad x = \frac{1}{4}$ or $x = 1$.
Now $f(1) = 1^4 - 3 \cdot 1^3 + 3 \cdot 1^2 - 1 = 0$ (neither maximum nor minimum) and
$f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^4 - 3\left(\frac{1}{4}\right)^3 + 3\left(\frac{1}{4}\right)^2 - \frac{1}{4} = -\frac{27}{256}$
(minimum). At the right endpoint we have
$f(2) = 2^4 - 3 \cdot 2^3 + 3 \cdot 2^2 - 2 = 2$ (maximum).

48. $f(x) = x^5$. $f'(x) = 5x^4 \quad \Rightarrow \quad f'(0) = 0$ so 0 is a critical number. But $f(0) = 0$ and f takes both positive and negative values in any open interval containing 0, so f has neither a local maximum nor a local minimum at 0.

49.