4.2 THE MEAN VALUE THEOREM

A Click here for answers.

S Click here for solutions.

1–4 ▪ Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

1. $f(x) = x^3 - x, \quad [-1, 1]$

2. $f(x) = x^3 + x^2 - 2x + 1, \quad [-2, 0]$

3. $f(x) = \cos 2x, \quad [0, \pi]$

4. $f(x) = \sin x + \cos x, \quad [0, 2\pi]$

5–11 ▪ Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

5. $f(x) = x^2 - 4x + 5, \quad [1, 5]$

6. $f(x) = x^3 - 2x + 1, \quad [-2, 3]$

7. $f(x) = 1 - x^2, \quad [0, 3]$

8. $f(x) = 2x^3 + x^2 - x - 1, \quad [0, 2]$

9. $f(x) = 1/x; \quad [1, 2]$

10. $f(x) = \sqrt{x}; \quad [1, 4]$

11. $f(x) = 1 + \sqrt[3]{x - 1}; \quad [2, 9]$

12. Verify that the function $f(x) = x^4 - 6x^3 + 4x - 1$ satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 1]$. Then use a graphing calculator or CAS to find, correct to two decimal places, the numbers c that satisfy the conclusion of the Mean Value Theorem.

13. Show that the equation $x^5 + 10x + 3 = 0$ has exactly one real root.

14. Show that the equation $3x - 2 + \cos(\pi x/2) = 0$ has exactly one real root.

15. Show that the equation $x^5 - 6x + c = 0$ has at most one root in the interval $[-1, 1]$.

16. Suppose f is continuous on $[2, 5]$ and $1 \leqslant f'(x) \leqslant 4$ for all x in $(2, 5)$. Show that $3 \leqslant f(5) - f(2) \leqslant 12$.

4.2 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $\pm\frac{1}{\sqrt{3}}$
2. $\frac{-1-\sqrt{7}}{3}$
3. $\frac{\pi}{2}$
4. $\frac{\pi}{4}, \frac{5\pi}{4}$
5. $\frac{3}{2}$
6. $\pm\sqrt{\frac{7}{3}}$
7. $\frac{3}{2}$
8. $\frac{-1+\sqrt{61}}{6}$
9. $\sqrt{2}$
10. $\frac{9}{4}$
11. $\left(\frac{7}{3}\right)^{3/2}+1$
12. $-\frac{1}{2}, \frac{5}{2} \pm \frac{1}{2}\sqrt{15}$

4.2 SOLUTIONS

E Click here for exercises.

1. $f(x) = x^3 - x$, $[-1, 1]$. f, being a polynomial, is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. Also $f(-1) = 0 = f(1)$. $f'(c) = 3c^2 - 1 = 0 \Rightarrow c = \pm\frac{1}{\sqrt{3}}$.

2. $f(x) = x^3 + x^2 - 2x + 1$, $[-2, 0]$. f, being a polynomial, is continuous on $[-2, 0]$ and differentiable on $(-2, 0)$. Also $f(-2) = 1 = f(0)$. $f'(c) = 3c^2 + 2c - 2 = 0 \Rightarrow c = \frac{-1\pm\sqrt{7}}{3}$, but only $\frac{-1-\sqrt{7}}{3}$ lies in the interval $(-2, 0)$.

3. $f(x) = \cos 2x$, $[0, \pi]$. f is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$. Also $f(0) = 1 = f(\pi)$. $f'(c) = -2\sin 2c = 0 \Rightarrow \sin 2c = 0 \Rightarrow 2c = \pi \Rightarrow c = \frac{\pi}{2}$ [since $c \in (0, \pi)$].

4. $f(x) = \sin x + \cos x$, $[0, 2\pi]$. Since $\sin x$ and $\cos x$ are continuous on $[0, 2\pi]$ and differentiable on $(0, 2\pi)$ so is their sum $f(x)$. $f(0) = 1 = f(2\pi)$. $f'(c) = \cos c - \sin c = 0 \Leftrightarrow \cos c = \sin c \Leftrightarrow c = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.

5. $f(x) = x^2 - 4x + 5$, $[1, 5]$. f, being a polynomial, is continuous on $[1, 5]$ and differentiable on $(1, 5)$. $\dfrac{f(5) - f(1)}{5 - 1} = \dfrac{10 - 2}{4} = 2$ and $2 = f'(c) = -2c \Rightarrow c = \frac{3}{2}$.

6. $f(x) = x^3 - 2x + 1$, $[-2, 3]$. f, being a polynomial, is continuous on $[-2, 3]$ and differentiable on $(-2, 3)$. $\dfrac{f(3) - f(-2)}{3 - (-2)} = \dfrac{22 - (-3)}{5} = 5$ and $5 = f'(c) = 3c^2 - 2 \Rightarrow 3c^2 = 7 \Rightarrow c = \pm\sqrt{\frac{7}{3}}$.

7. $f(x) = 1 - x^2$, $[0, 3]$. f, being a polynomial, is continuous on $[0, 3]$ and differentiable on $(0, 3)$. $\dfrac{f(3) - f(0)}{3 - 0} = \dfrac{-8 - 1}{3} = -3$ and $-3 = f'(c) = -2c \Rightarrow c = \frac{3}{2}$.

8. $f(x) = 2x^3 + x^2 - x - 1$, $[0, 2]$. f, being a polynomial, is continuous on $[0, 2]$ and differentiable on $(0, 2)$. $\dfrac{f(2) - f(0)}{2 - 0} = \dfrac{17 - (-1)}{2} = 9$ and $9 = f'(c) = 6c^2 + 2c - 1 \Rightarrow 0 = 6c^2 + 2c - 10 \Rightarrow c = \frac{-2\pm\sqrt{244}}{2} = \frac{-1\pm\sqrt{61}}{6}$, but only $\frac{-1+\sqrt{61}}{6}$ lies in $(0, 2)$.

9. $f(x) = 1/x$, $[1, 2]$. f, being a rational function, is continuous on $[1, 2]$ and differentiable on $(1, 2)$. $\dfrac{f(2) - f(1)}{2 - 1} = \dfrac{\frac{1}{2} - 1}{1} = -\frac{1}{2}$ and $-\frac{1}{2} = f'(c) = -\frac{1}{c^2} \Rightarrow c^2 = 2 \Rightarrow c = \sqrt{2}$ (since c must lie in $[1, 2]$).

10. $f(x) = \sqrt{x}$, $[1, 4]$. $f(x)$ is continuous on $[1, 4]$ and differentiable on $(1, 4)$. $\dfrac{f(4) - f(1)}{4 - 1} = \dfrac{2 - 1}{3} = \dfrac{1}{3}$ and $\dfrac{1}{3} = f'(c) = \dfrac{1}{2\sqrt{c}} \Rightarrow \sqrt{c} = \frac{3}{2} \Rightarrow c = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

11. 1, $x - 1$, and $\sqrt[3]{x}$ are continuous on $\mathbb{R}$, and therefore $f(x) = 1 + \sqrt[3]{x - 1}$ is continuous on $\mathbb{R}$, and hence continuous on $[2, 9]$. $f'(x) = \frac{1}{3}(x - 1)^{-2/3}$, so that f is differentiable for all $x \neq 1$ and so f is differentiable on $(2, 9)$. By the Mean Value Theorem, there exists a number c such that $f'(c) = \frac{1}{3}(c - 1)^{-2/3} = \dfrac{f(9) - f(2)}{9 - 2} = \dfrac{3 - 2}{7} = \dfrac{1}{7} \Rightarrow \frac{1}{3}(c - 1)^{-2/3} = \frac{1}{7} \Rightarrow (c - 1)^2 = \left(\frac{7}{3}\right)^3 \Rightarrow c = \pm\left(\frac{7}{3}\right)^{3/2} + 1 \Rightarrow c = \left(\frac{7}{3}\right)^{3/2} + 1 \approx 4.564$ since $c \in [2, 9]$.

12. $f(x) = x^4 - 6x^3 + 4x - 1$ is continuous and differentiable on $\mathbb{R}$ since it is a polynomial. So by the Mean Value Theorem there exists a number c such that

$$\begin{aligned} f'(c) &= 4c^3 - 18c^2 + 4 = \frac{f(1) - f(0)}{1 - 0} \\ &= \frac{-2 - (-1)}{1} = -1 \end{aligned}$$

$\Rightarrow 4c^3 - 18c^2 + 5 = 0$. Using a CAS to solve this equation, we find that the numbers, correct to two decimal places, are $c = 4.44, 0.56$, and -0.50.

13. $f(x) = x^5 + 10x + 3$. Since f is continuous and $f(-1) = -8$ and $f(0) = 3$, the equation $f(x) = 0$ has at least one root in $(-1, 0)$ by the Intermediate Value Theorem. Suppose that the equation has more than one root; say a and b are both roots with $a < b$. Then $f(a) = 0 = f(b)$ so by Rolle's Theorem $f'(x) = 5x^4 + 10 = 0$ has a root in (a, b). But this is impossible since clearly $f'(x) \geq 10 > 0$ for all real x.

14. $f(x) = 3x - 2 + \cos\left(\frac{\pi}{2}x\right)$. Since f is continuous and $f(0) = -1$ and $f(1) = 1$, the equation $f(x) = 0$ has at least one root in $(0, 1)$ by the Intermediate Value Theorem. Suppose it has more than one root; say $a < b$ are both roots. Then $f(a) = 0 = f(b)$, so by Rolle's Theorem, $f'(x) = 3 - \frac{\pi}{2}\sin\left(\frac{\pi}{2}x\right) = 0$ has a root in (a, b). But this is impossible since $-\sin x \geq -1 \Rightarrow f'(x) \geq 3 - \frac{\pi}{2} > 0$ for all real x.

15. $f(x) = x^5 - 6x + c$. Suppose that $f(x) = 0$ has two roots a and b with $-1 \le a < b \le 1$. Then $f(a) = 0 = f(b)$, so by Rolle's Theorem there is a number d in (a, b) with $f'(d) = 0$. Now $0 = f'(d) = 5d^4 - 6 \quad \Rightarrow \quad d = \pm\sqrt[4]{\frac{6}{5}}$, which are both outside $[-1, 1]$ and hence outside (a, b). Thus, $f(x)$ can have at most one root in $[-1, 1]$.

16. By the Mean Value Theorem, $\dfrac{f(5) - f(2)}{5 - 2} = f'(c)$ for some $c \in (2, 5)$. Since $1 \le f'(x) \le 4$, we have $1 \le \dfrac{f(5) - f(2)}{5 - 2} \le 4$ or $1 \le \dfrac{f(5) - f(2)}{3} \le 4$ or $3 \le f(5) - f(2) \le 12$.