4.5 OPTIMIZATION PROBLEMS

A Click here for answers.

S Click here for solutions.

1. Find the points on the hyperbola $y^2 - x^2 = 4$ that are closest to the point (2, 0).

2. Find the point on the parabola $x + y^2 = 0$ that is closest to the point $(0, -3)$.

3–8 ■ For each cost function (given in dollars), find (a) the cost, average cost, and marginal cost at a production level of 1000 units; (b) the production level that will minimize the average cost; and (c) the minimum average cost.

3. $C(x) = 10{,}000 + 25x + x^2$

4. $C(x) = 1600 + 8x + 0.01x^2$

5. $C(x) = 45 + \dfrac{x}{2} + \dfrac{x^2}{560}$

6. $C(x) = 2000 + 10x + 0.001x^3$

7. $C(x) = 2\sqrt{x} + x^2/8000$

8. $C(x) = 1000 + 96x + 2x^{3/2}$

4.5 ANSWERS

E Click here for exercises.

S Click here for solutions.

1. $(1, \pm\sqrt{5})$
2. $(-1, -1)$
3. (a) \$1,035,000, \$1035/unit, \$2025/unit.
   (b) 100
   (c) \$225/unit
4. (a) \$19,600, \$19.60/unit, \$28/unit
   (b) 400
   (c) \$16/unit
5. (a) \$2330.71, \$2.33/unit, \$4.07/unit
   (b) 159
   (c) \$1.07/unit
6. (a) \$1,012,000, \$1012/unit, \$3010/unit
   (b) 100
   (c) \$40/unit
7. (a) \$188.25, \$0.19/unit, \$0.28/unit
   (b) 400
   (c) \$0.15/unit
8. (a) \$160,245.55, \$160.25/unit, \$190.87/unit
   (b) 100
   (c) \$126/unit

4.5 SOLUTIONS

E Click here for exercises.

1. By symmetry, the points are (x, y) and $(x, -y)$, where $y > 0$. The square of the distance is

$$\begin{aligned} D(x) &= (x-2)^2 + y^2 \\ &= (x-2)^2 + (4 + x^2) \\ &= 2x^2 - 4x + 8 \end{aligned}$$

So $D'(x) = 4x - 4 = 0 \quad \Rightarrow \quad x = 1$ and $y = \pm\sqrt{4+1} = \pm\sqrt{5}$. The points are $(1, \pm\sqrt{5})$.

2. The square of the distance from a point (x, y) on the parabola $x = -y^2$ is $x^2 + (y+3)^2 = y^4 + y^2 + 6y + 9 = D(y)$. Now

$$\begin{aligned} D'(y) &= 4y^3 + 2y + 6 \\ &= 2(y+1)(2y^2 - 2y + 3) \end{aligned}$$

Since $2y^2 - 2y + 3 = 0$ has no real roots, $y = -1$ is the only critical number. Then $x = -(-1)^2 = -1$, so the point is $(-1, -1)$.

3. (a) $C(x) = 10{,}000 + 25x + x^2$, $C(1000) = \$1{,}035{,}000$,

$c(x) = \dfrac{C(x)}{x} = \frac{10{,}000}{x} + 25 + x$, $c(1000) = \$1035$.

$C'(x) = 25 + 2x$, $C'(1000) = \$2025/\text{unit}$.

(b) We must have $c(x) = C'(x) \quad \Rightarrow$ $10{,}000/x + 25 + x = 25 + 2x \quad \Rightarrow \quad 10{,}000/x = x$ $\Rightarrow \quad x^2 = 10{,}000 \quad \Rightarrow \quad x = 100$. This is a minimum since $c''(x) = 20{,}000/x^3 > 0$.

(c) The minimum average cost is $c(100) = \$225/\text{unit}$.

4. (a) $C(x) = 1600 + 8x + 0.01x^2$, $C(1000) = \$19{,}600$.

$c(x) = \dfrac{1600}{x} + 8 + 0.01x$, $c(1000) = \$19.60$.

$C'(x) = 8 + 0.02x$, $C'(1000) = \$28$.

(b) We must have $C'(x) = c(x) \quad \Leftrightarrow$

$8 + 0.02x = \dfrac{1600}{x} + 8 + 0.01x \quad \Leftrightarrow \quad 0.01x = \dfrac{1600}{x}$

$\Leftrightarrow \quad x^2 = \dfrac{1600}{0.01} = 160{,}000 \quad \Leftrightarrow \quad x = 400$. This is a minimum since $c''(x) = 3200/x^3 > 0$ for $x > 0$.

(c) The minimum average cost is $c(400) = \$16/\text{unit}$.

5. (a) $C(x) = 45 + \dfrac{x}{2} + \dfrac{x^2}{560}$, $C(1000) = \$2330.71$.

$c(x) = \dfrac{45}{x} + \dfrac{1}{2} + \dfrac{x}{560}$, $c(1000) = \$2.33/\text{unit}$.

$C'(x) = \dfrac{1}{2} + \dfrac{x}{280}$, $C'(1000) = \$4.07/\text{unit}$.

(b) We must have $C'(x) = c(x) \quad \Rightarrow$

$\dfrac{1}{2} + \dfrac{x}{280} = \dfrac{45}{x} + \dfrac{1}{2} + \dfrac{x}{560} \quad \Rightarrow \quad \dfrac{45}{x} = \dfrac{x}{560} \quad \Rightarrow$ $x^2 = (45)(560) \quad \Rightarrow \quad x = \sqrt{25{,}200} \approx 159$. This is a minimum since $c''(x) = 90/x^2 > 0$.

(c) The minimum average cost is $c(159) = \$1.07/\text{unit}$.

6. (a) $C(x) = 2000 + 10x + 0.001x^3$, $C(1000) = \$1{,}012{,}000$.

$c(x) = \dfrac{2000}{x} + 10 + 0.001x^2$, $c(1000) = \$1012/\text{unit}$.

$C'(x) = 10 + 0.003x^2$, $C'(1000) = \$3010/\text{unit}$.

(b) We must have $C'(x) = c(x) \quad \Leftrightarrow$

$10 + 0.003x^2 = \dfrac{2000}{x} + 10 + 0.001x^2 \quad \Leftrightarrow$

$\dfrac{2000}{x} = 0.002x^2 \quad \Leftrightarrow \quad x^3 = 2000/0.002 = 1{,}000{,}000$

$\Leftrightarrow \quad x = 100$. This is a minimum since $c''(x) = \dfrac{4000}{x^3} + 0.002 > 0$ for $x > 0$.

(c) The minimum average cost is $c(100) = \$40/\text{unit}$.

7. (a) $C(x) = 2\sqrt{x} + \dfrac{x^2}{8000}$, $C(1000) = \$188.25$.

$c(x) = \dfrac{2}{\sqrt{x}} + \dfrac{x}{8000}$, $c(1000) = \$0.19/\text{unit}$.

$C'(x) = \dfrac{1}{\sqrt{x}} + \dfrac{x}{4000}$, $C'(1000) = \$0.28/\text{unit}$.

(b) We must have $C'(x) = c(x) \quad \Rightarrow$

$\dfrac{1}{\sqrt{x}} + \dfrac{x}{4000} = \dfrac{2}{\sqrt{x}} + \dfrac{x}{8000} \quad \Rightarrow \quad \dfrac{x}{8000} = \dfrac{1}{\sqrt{x}} \quad \Rightarrow$ $x^{3/2} = 8000 \quad \Rightarrow \quad x = (8000)^{2/3} = 400$. This is a minimum since $c''(x) = \frac{3}{2}x^{-5/2} > 0$.

(c) The minimum average cost is $c(400) = \$0.15/\text{unit}$.

8. (a) $C(x) = 1000 + 96x + 2x^{3/2}$, $C(1000) = \$160{,}245.55$.

$c(x) = \dfrac{1000}{x} + 96 + 2\sqrt{x}$, $c(1000) = \$160.25/\text{unit}$.

$C'(x) = 96 + 3\sqrt{x}$, $C'(1000) = \$190.87/\text{unit}$.

(b) We must have $C'(x) = c(x) \quad \Leftrightarrow$

$96 + 3\sqrt{x} = 1000/x + 96 + 2\sqrt{x} \quad \Leftrightarrow \quad \sqrt{x} = 1000/x$

$\Leftrightarrow \quad x^{3/2} = 1000 \quad \Leftrightarrow \quad x = (1000)^{2/3} = 100$. Since $c'(x) = \left(x^{3/2} - 1000\right)/x^2 < 0$ for $0 < x < 100$ and $c'(x) > 0$ for $x > 100$, there is an absolute minimum at $x = 100$.

(c) The minimum average cost is $c(100) = \$126/\text{unit}$.